<u>§1 Review of Governing Equations for Water Flow Dynamics.</u>

> Derivation of Governing Eq. for Transport Phenomena Caused by Flow.

§1 Review of Governing Equations for Water Flow Dynamics.

Flow of Water is Basically Described by Navier-Stokes (NS) Eq.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right)-g$$

u, v, w : Flow Velocity Component in x, y, z-direction.

ρ : Density of Water. P : Pressure. ν : Kinematic Viscosity Coefficient.

NS Eq. : Equation of Motion of Water Body

⇒ Time Development of Spatial Distribution of Flow Velocity u, v, w can be Known by Solving NS Eq.

To Solve the NS Eq.

Pressure P & Density ρ must be Determined.

NS Eq. is Solved with Supplemental Eqs. to Describe the change of P and ρ .

1: Continuous Equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Derived from Incompressibility of Water.

Pressure is Determined to Satisfy This Eq.

2: Transport Equation

$$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x}+v\frac{\partial \phi}{\partial y}+w\frac{\partial \phi}{\partial z}=\frac{\partial}{\partial x}\left(\nu\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial \phi}{\partial z}\right)$$

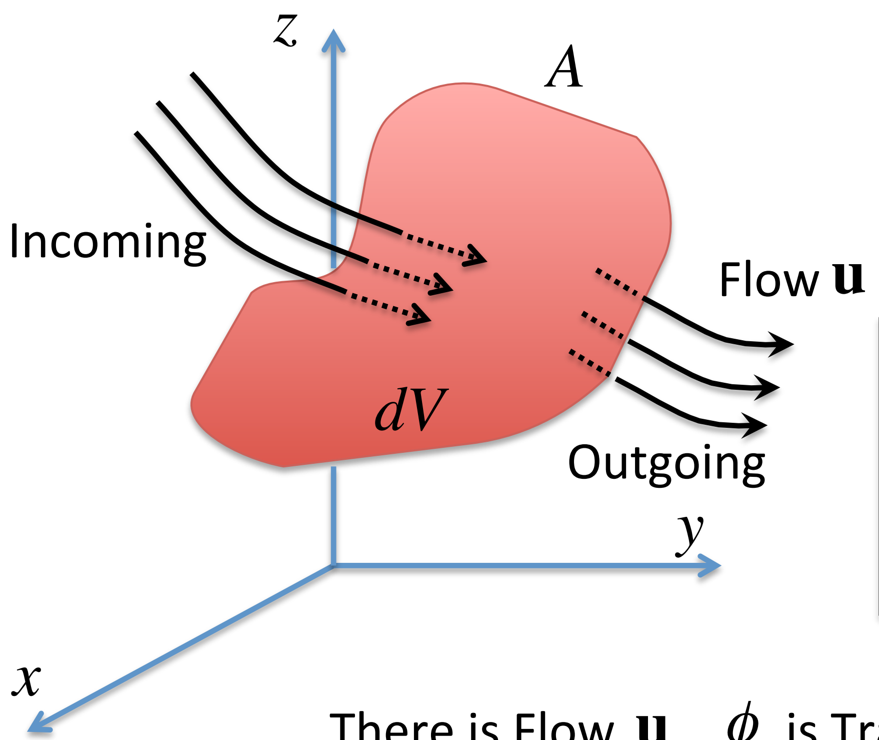
ϕ:Scalar Quantity (Density ρ ,Salinity, Water Temp.,....)

§1 Deviation of the Governing Equation

§1.1 General Form of Transport Equation

$\phi(t,x,y,z)$:Amount of Arbitrary Substance Included per Unit Volume.
(Such as Density ρ)

Small Element with Arbitrary Shape in Fluid



Total Surface Area: A

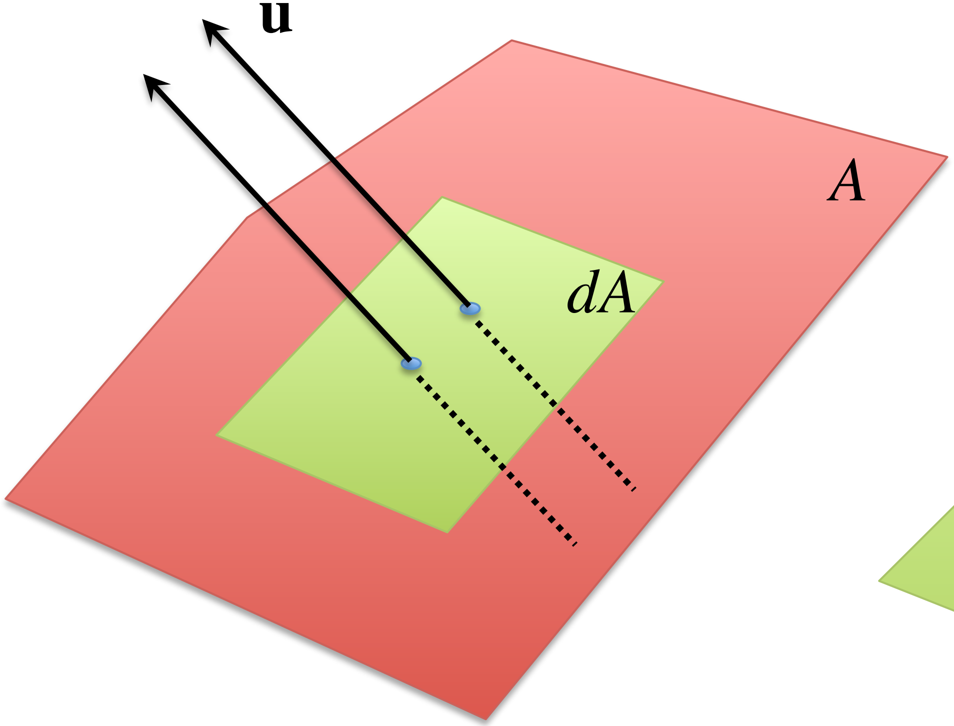
Total Volume: dV

Total Amount of ϕ Included in dV ;

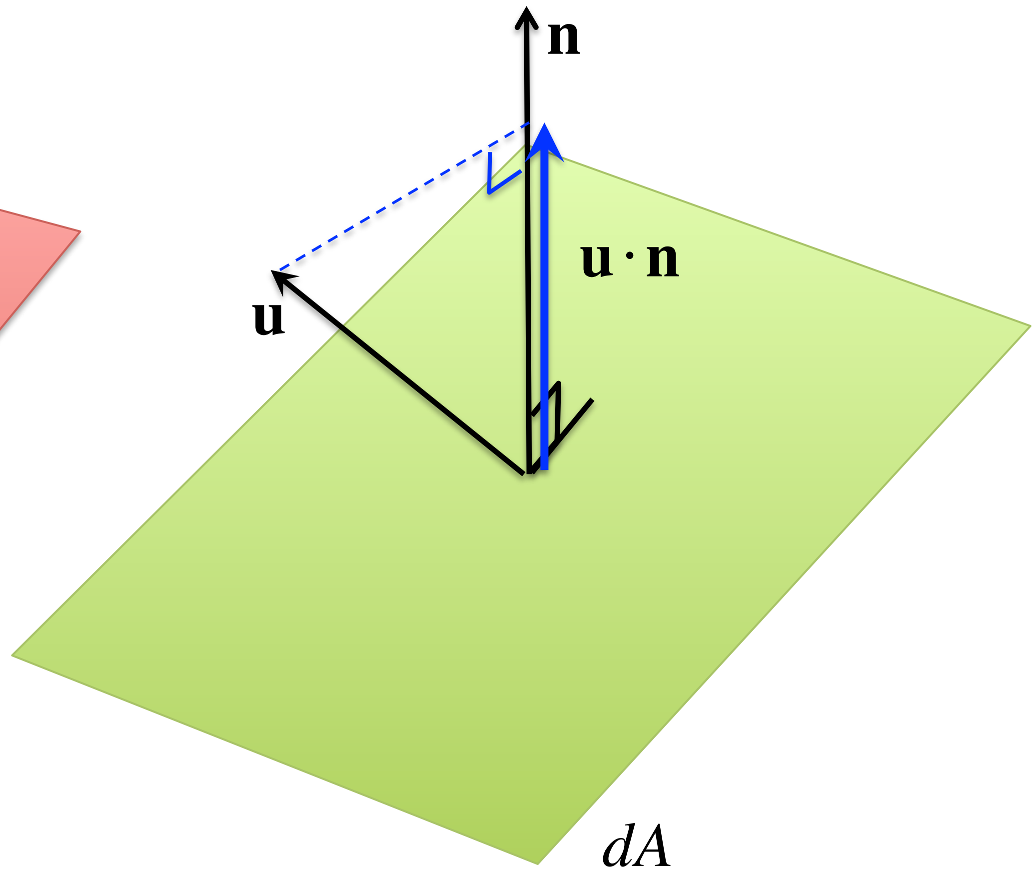
$$M(t) = \int \phi(t,x,y,z)dV$$

There is Flow $\mathbf{u}$, ϕ is Transported Through The Surface A.

Small Area dA on A .



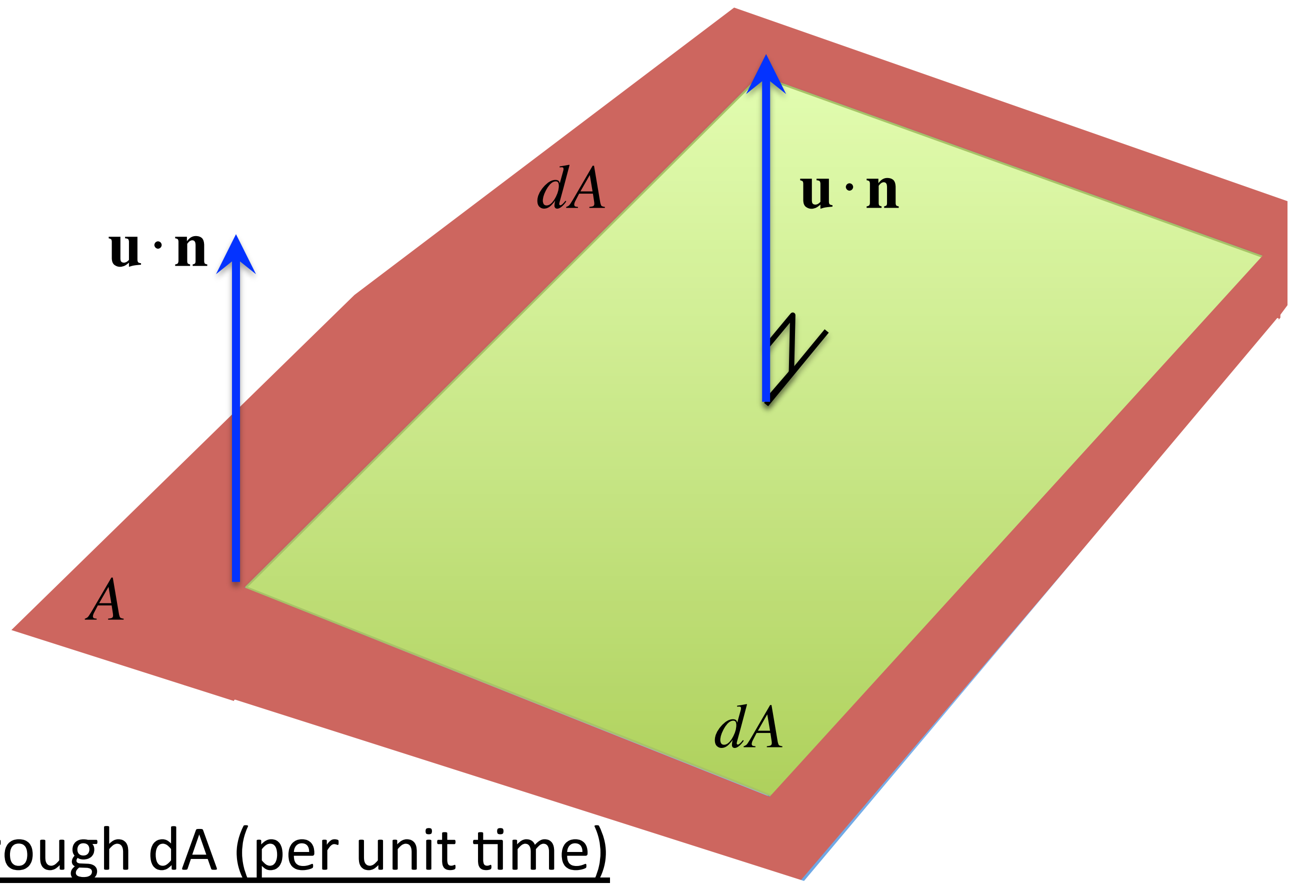
Outgoing Through dA



$\mathbf{n}$: Normal Unit Vector to dA .

Normal Component of Flow Velocity : $\mathbf{u} \cdot \mathbf{n}$
(Outward is Positive)

Small Area dA on A .



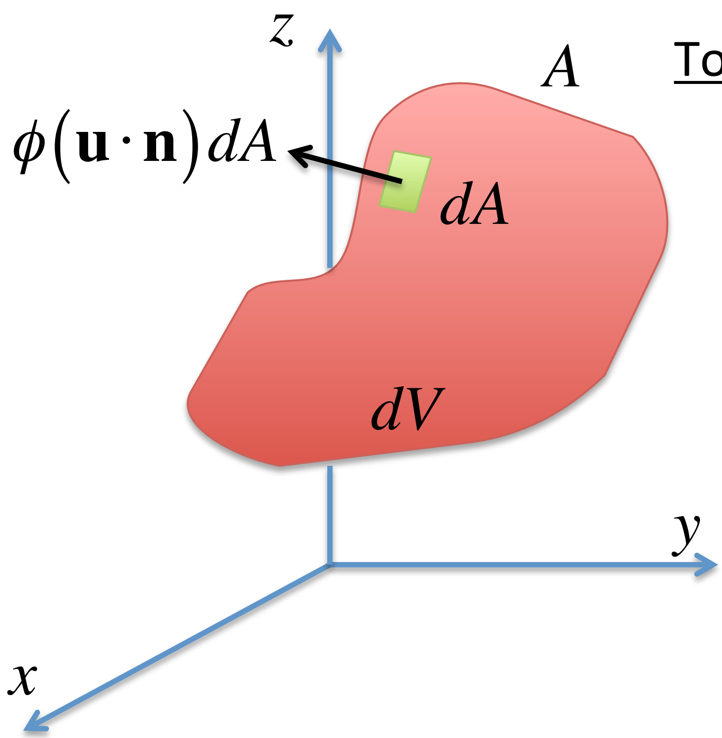
Outgoing Volume Through dA (per unit time)

$$\left(\mathbf{u}\cdot\mathbf{n}\right)\times dA$$

Outgoing Amount of ϕ Through dA (per unit time)

$$\phi\times\left(\mathbf{u}\cdot\mathbf{n}\right)\times dA$$

Total Amount of Transportation of ϕ Through Whole Surface



Total Amount of ϕ Outgoing Through Whole Surface

$$F = \oint_A \phi(\mathbf{u}\cdot\mathbf{n})dA$$

Total Amount of ϕ Included in dV ;

$$M(t) = \int \phi(t,x,y,z)dV$$

If there is not any production in dV, Conservation of ϕ must be Satisfied;

"Change of Total Amount included in dV (M)" = "Total Amount of Outgoing(F)"

$$\frac{\partial M}{\partial t} = -F$$

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$$\frac{\partial M}{\partial t} = -F$$

$$\frac{\partial}{\partial t}\left(\int \phi(t,x,y,z)\,dV \right) = -\oint_A \phi\left(\mathbf{u}\cdot\mathbf{n}\right)dA$$

$$\therefore \frac{\partial}{\partial t}\left(\int \phi(t,x,y,z)\,dV \right) + \oint_A \phi\left(\mathbf{u}\cdot\mathbf{n}\right)dA = 0$$

<u>Gauss's Divergence Theorem</u>

$$\oint_A \left(\mathbf{F}\cdot\mathbf{n}\right)dA = \int \nabla\cdot\mathbf{F}\,dV = \int \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

$\mathbf{F} = (F_x, F_y, F_z)$: Arbitrary Vector

<u>Surface Integration can be Transformed in Volume Integration.</u>

By Using Gauss's Theorem,

$$\frac{\partial}{\partial t}\left(\int \phi(t,x,y,z)\,dV\right)+\oint_A \phi\left(\mathbf{u}\cdot\mathbf{n}\right)dA=0$$

Regarding $\phi\mathbf{u}$ as $\mathbf{F}$.

$$\frac{\partial}{\partial t}\left(\int \phi(t,x,y,z)\,dV\right)+\int \nabla\cdot\left(\phi\mathbf{u}\right)dV=0$$

Change of Order

$$\left(\int \frac{\partial}{\partial t}\phi\,dV\right)+\int \nabla\cdot\left(\phi\mathbf{u}\right)dV=0$$

$$\therefore \int\left(\frac{\partial}{\partial t}\phi+\nabla\cdot\left(\phi\mathbf{u}\right)\right)dV=0$$

Integrand

This Relation must be Satisfied for Arbitrary Shape dV .

➡ Integrand must be "0".

As a result, from the Conservation Law,

Governing Eq. Described the Change Due to Flow;

$$\frac{\partial}{\partial t}\phi + \nabla \cdot (\phi \mathbf{u}) = 0$$

(Vector form)

$$\frac{\partial \phi}{\partial t} + \frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y} + \frac{\partial w\phi}{\partial z} = 0$$

(non-Vector form)

(Conservative Equation)