§1.2 Physical Meaning of Transport Eq.

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$$\frac{\partial \phi}{\partial t} + \frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y} + \frac{\partial w\phi}{\partial z} = 0$$

Expanding the Differentials

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} = -\phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

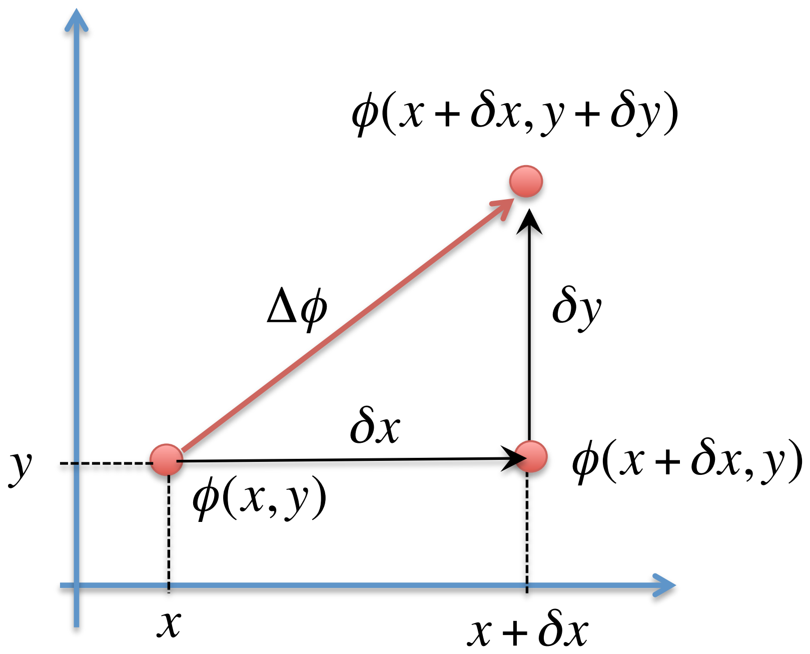
(Conservative Equation in Eulerian Form)

Investigate Physical Meaning of Each Terms

L.H.S : $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}$

Chain Rule for Multi-Variable Function

Multi-Variable Function (depends on two variables, x and y) : $\phi = \phi(x, y)$



Change Due to a shift in x-direction with distance δx.

$$A = \phi(x+\delta x, y) - \phi(x, y) = \frac{\partial \phi}{\partial x} \times \delta x$$

Change Due to a shift in y-direction with distance δy

$$B = \phi(x+\delta x, y+\delta y) - \phi(x+\delta x, y) = \frac{\partial \phi}{\partial y} \times \delta y$$

Change of ϕ Between $(x+\delta x, y+\delta y)$ and (x, y).

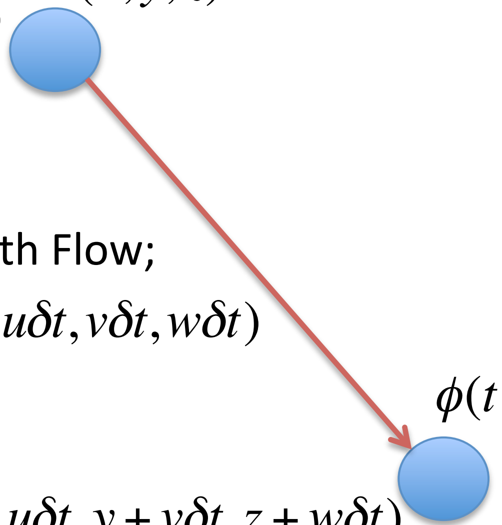
$$\Delta\phi = \phi(x+\delta x, y+\delta y) - \phi(x, y) = A + B$$

$$\therefore \Delta\phi = \delta x \frac{\partial \phi}{\partial x} + \delta y \frac{\partial \phi}{\partial y}$$

Total Change of Multi-Variable Function is given by Summation of Gradient in Each Direction (Chain Rule).

Considering a Small Element of Water Moving with Flow Velocity $\mathbf{u} = (u, v, w)$

$\phi(t, x, y, z)$ (x, y, z)



Assuming, Initial Position is (x, y, z) .

During the Period of δt ,

Element Moves to $(x + u\delta t, y + v\delta t, z + w\delta t)$.

Move with Flow;

$\mathbf{u} \times \delta t = (u\delta t, v\delta t, w\delta t)$

$\phi(t + \delta t, x + u\delta t, y + v\delta t, z + w\delta t)$

$(x + u\delta t, y + v\delta t, z + w\delta t)$

Change of ϕ on the Moving Element during δt .

$$\phi(t + \delta t, x + u\delta t, y + v\delta t, z + w\delta t) - \phi(t, x, y, z) = \delta t \frac{\partial \phi}{\partial t} + u\delta t \frac{\partial \phi}{\partial x} + v\delta t \frac{\partial \phi}{\partial y} + w\delta t \frac{\partial \phi}{\partial z}$$

(Chain Rule)

Dividing by δt ,

$$\therefore \frac{\phi(t+\delta t, x+u\delta t, y+v\delta t, z+w\delta t)-\phi(t,x,y,z)}{\delta t} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}$$

Change of ϕ in Moving with Water Flow per Unit Time.

In the Hydraulics, a special symbol to represent Change of ϕ in moving with water flow.

Lagrangian Derivative : $\dfrac{D\phi}{Dt} = \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} + v\dfrac{\partial \phi}{\partial y} + w\dfrac{\partial \phi}{\partial z}$

Conservative Eq. in Eulerian Form

$$\underline{\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}} = -\phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \frac{D\phi}{Dt}$$ Lagrangian Derivative

$$\therefore \frac{D\phi}{Dt} = -\phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\phi\left(\nabla\cdot\mathbf{u}\right)$$

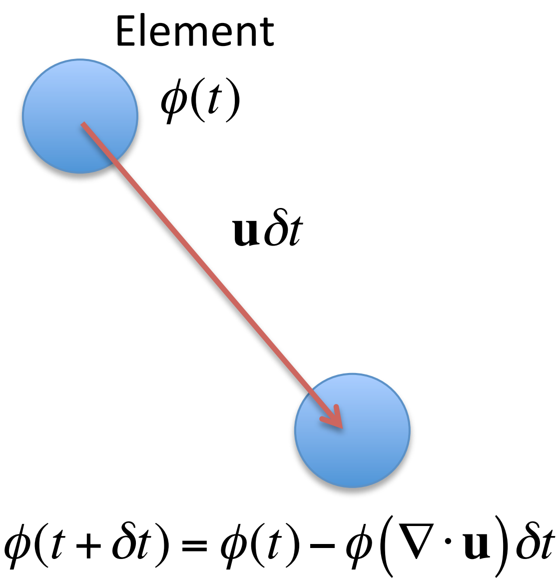
Physical Meaning of Conservative Eq.

ϕ on the Element is changing according to $-\phi\left(\nabla\cdot\mathbf{u}\right)$.

After Moving during δt ,

$$\phi(t+\delta t) = \phi(t) - \phi\left(\nabla\cdot\mathbf{u}\right)\delta t$$

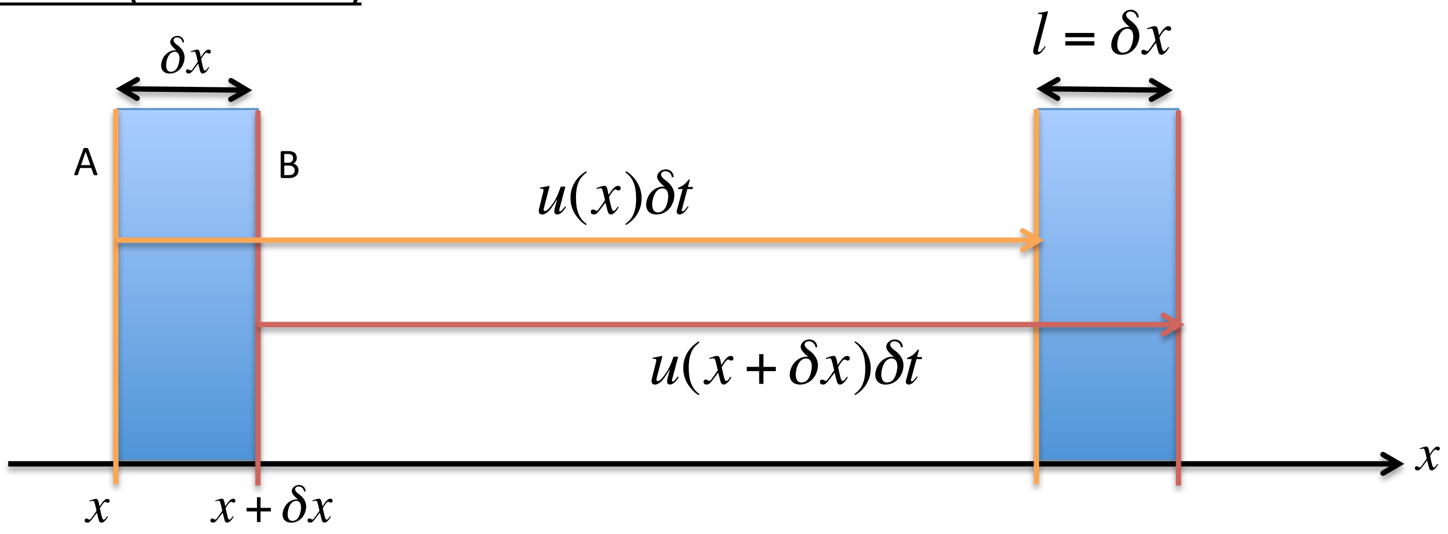
Physical Meaning of $-\phi\left(\nabla\cdot\mathbf{u}\right)$?



Physical Meaning of $-\phi(\nabla\cdot\mathbf{u})$?

1D Conservative Eq. : $\dfrac{D\phi}{Dt} = -\phi\dfrac{\partial u}{\partial x}$ "Assuming v=w=0"

<u>Small Element (width δx)</u>



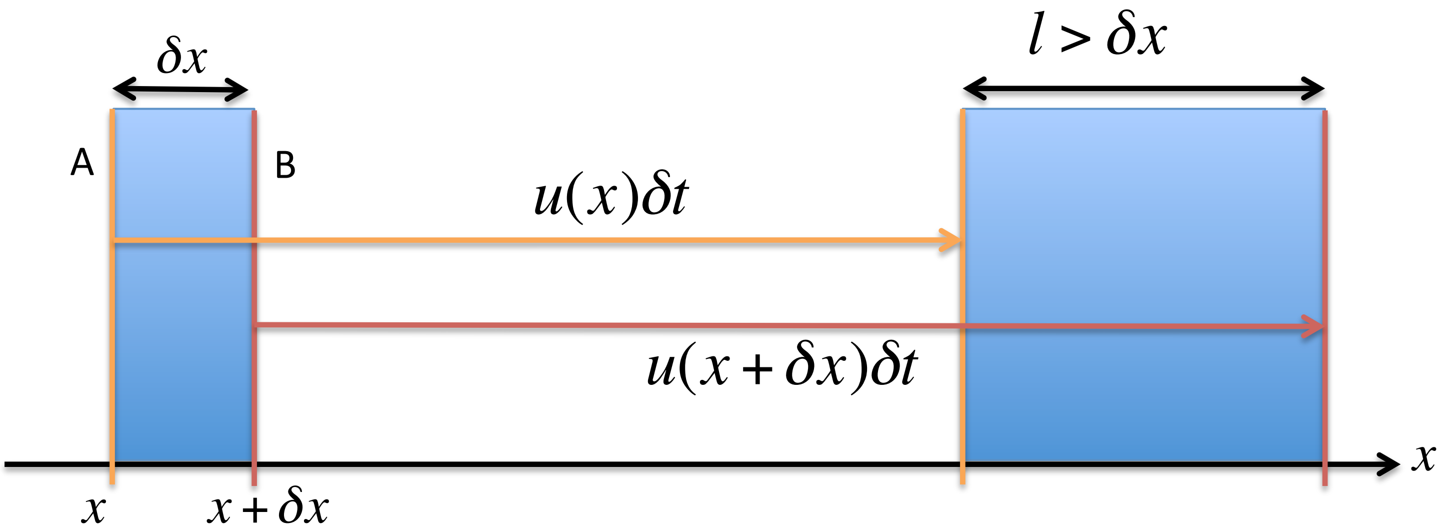
Case1: Spatial Gradient in x is "0"; $\dfrac{\partial u}{\partial x} = 0$

"Velocity at A" : $u(x)$ **=** "Velocity at B" : $u(x+\delta x)$

>A and B Boundary Move the Same Distance: $u(x)\delta t = u(x+\delta x)\delta t$

>Width After Moving : $l = \delta x$

Volume of Element is Not Changed!



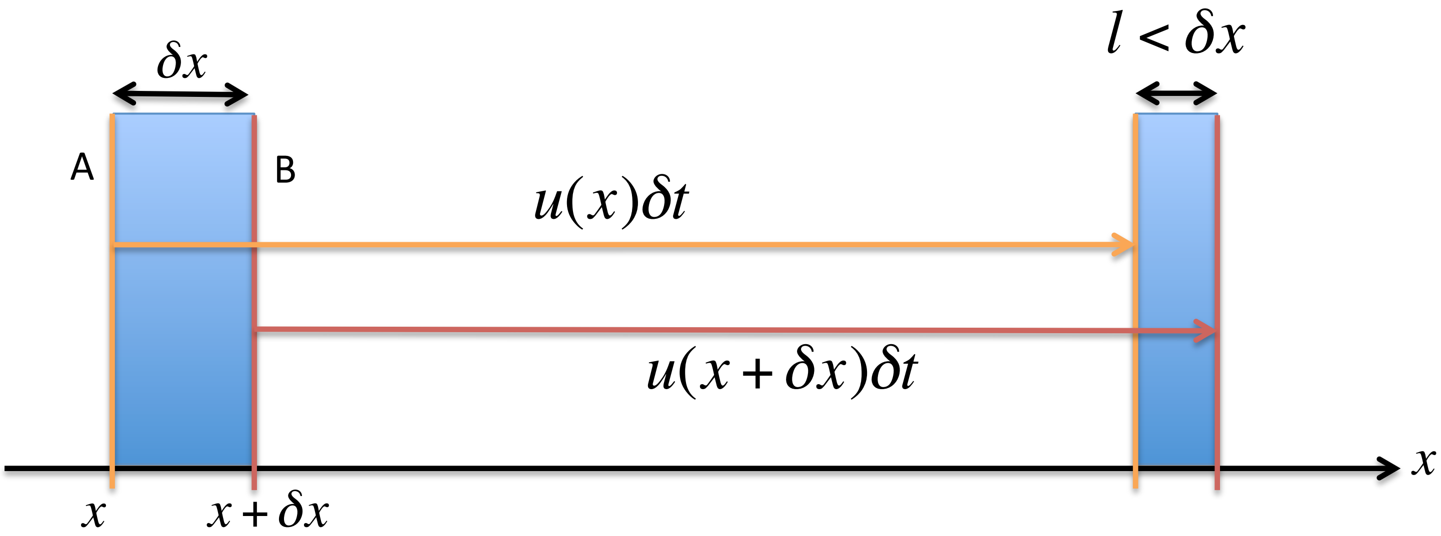
Case2: Spatial Gradient in x is Positive; $\frac{\partial u}{\partial x} > 0$

"Velocity at A" : $u(x)$ $<$ "Velocity at B" : $u(x+\delta x)$

>B Boundary Move the Longer Distance than A: $u(x)\delta t < u(x+\delta x)\delta t$

>Width After Moving : $l > \delta x$

Volume of Element is Expanded!



Case3: Spatial Gradient in x is Negative; $\frac{\partial u}{\partial x} < 0$

"Velocity at A" : $u(x)$ > "Velocity at B" : $u(x+\delta x)$

>B Boundary Move the Shorter Distance than A: $u(x)\delta t > u(x+\delta x)\delta t$

>Width After Moving : $l < \delta x$

Volume of Element is Compressed!

Compressed or Expanded? ...Depends on Spatial Gradient

When

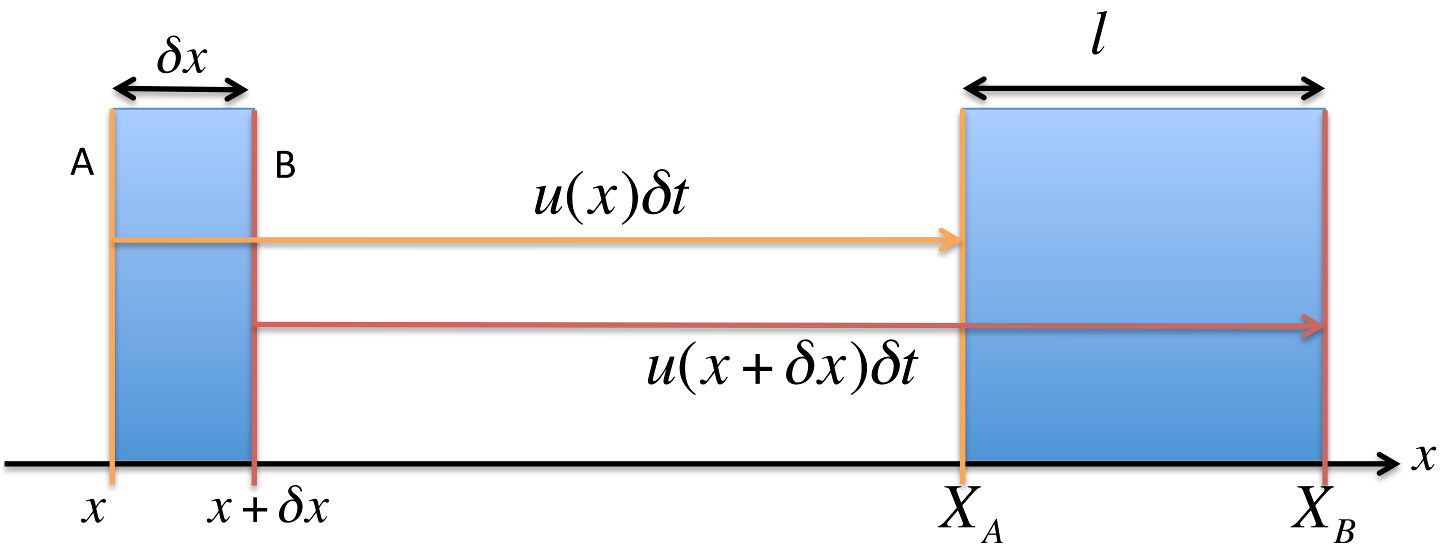
$$\frac{\partial u}{\partial x} = 0$$ → Volume is not Changed (Not Compressed and Expanded).

$$\frac{\partial u}{\partial x} > 0$$ → Expanded

$$\frac{\partial u}{\partial x} < 0$$ → Compressed

Change of ϕ Due to the Volume Change ?

Change of ϕ Due to the Volume Change ?



>"Velocity at B" $: u(x+\delta x) = u(x) + \dfrac{\partial u}{\partial x}\delta x$

>Position of Each Boundary After δt;

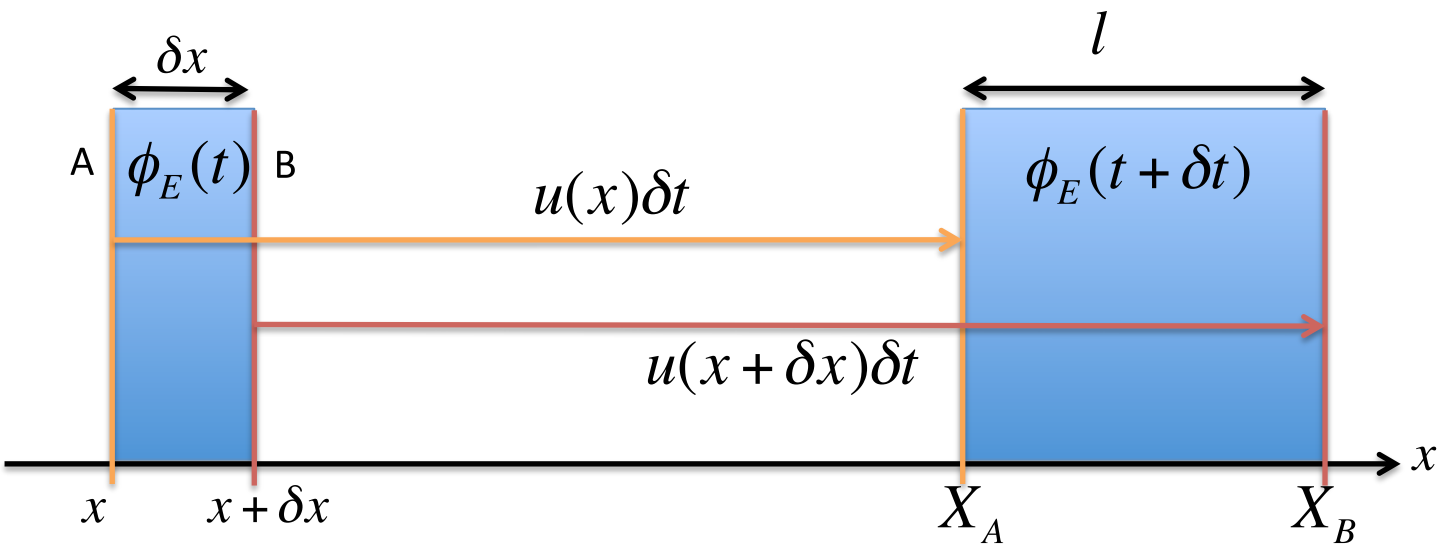
A: $X_A = x + u(x)\delta t$

B: $X_B = \left(x+\delta x\right) + u(x+\delta x)\delta t = \left(x+\delta x\right) + \left(u(x) + \dfrac{\partial u}{\partial x}\delta x\right)\delta t$

>Width After δt;

$$l = X_B - X_A = \left(x+\delta x\right) + \left(u(x) + \frac{\partial u}{\partial x}\delta x\right)\delta t - \left(x + u(x)\delta t\right)$$

Change of ϕ Due to the Volume Change ?



>Width After δt; $l = X_B - X_A = \delta x + \delta x \delta t \dfrac{\partial u}{\partial x}$

>Assuming ϕ on the Element; $\phi_E(t)$

>Total Amount of ϕ included in Element;

"Before Moving; $\phi_E(t) \times \delta x$" $=$ "After Moving; $\phi_E(t+\delta t) \times l$ "

Total Amount included in Elem. Must not be Changed Through Moving.

Change of ϕ Due to the Volume Change ?

"Before Moving; $\phi_E(t)\times\delta x$" **=** "After Moving; $\phi_E(t+\delta t)\times l$ "

$$\phi_E(t)\delta x = \phi_E(t+\delta t)l$$

Substituting

$$l = \delta x + \delta x\delta t\frac{\partial u}{\partial x}$$

$$\phi_E(t)\delta x = \phi_E(t+\delta t)\times\left(\delta x + \delta x\delta t\frac{\partial u}{\partial x}\right)$$

Divided by δx

$$\phi_E(t) = \phi_E(t+\delta t)\times\left(1 + \delta t\frac{\partial u}{\partial x}\right)$$

Divided by δt

$$\frac{\phi_E(t+\delta t) - \phi_E(t)}{\delta t} = -\phi_E(t+\delta t)\times\frac{\partial u}{\partial x}$$

Taking a limit $\delta t \to 0$;

$$\therefore \frac{D\phi_E}{Dt} = -\phi_E(t)\times\frac{\partial u}{\partial x}$$

Change of ϕ Due to the Volume Change ?

As a result, we can get "1D Conservative Eq."

$$\therefore \frac{D\phi_E}{Dt} = -\phi_E(t) \times \frac{\partial u}{\partial x}$$

R.H.S of "1D Conservative Eq."; $-\phi_E(t) \times \frac{\partial u}{\partial x}$

→ Means the Affect of Compression or Expansion.

In 3D case, Same Result can be get Straightforwardly,

Instead of $\frac{\partial u}{\partial x}$,

Divergence of Velocity $\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ indicates Tendency of "Volume Change".

- $\nabla \cdot \mathbf{u} = 0$ → Not Compressed and Expanded
- $\nabla \cdot \mathbf{u} > 0$ → Expanded
- $\nabla \cdot \mathbf{u} < 0$ → Compressed