§2.2 Governing Equation Specialized for Flow with Density Change – Boussinesq Eq.

Fundamental Eqs. for Flow with Spatial Density Change

Motion of Flow

- N.S Eq. ; $$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{u} + \mathbf{F}_G \qquad (2\text{-}1)$$
- Continuity Eq.; $$\nabla \cdot \mathbf{u} = 0 \qquad (2\text{-}2)$$

Diffusion Term; $$\nu\Delta\mathbf{u} \equiv \frac{\partial^2\mathbf{u}}{\partial x^2} + \frac{\partial^2\mathbf{u}}{\partial y^2} + \frac{\partial^2\mathbf{u}}{\partial z^2}$$ Δ; Symbol of Laplacian

External Force : $$\mathbf{F}_G = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$ Only Gravity (Ignore Friction Forces)

+

Transport Eq. for Density $$\frac{D\rho}{Dt} = \nu\Delta\rho \qquad (2\text{-}3)$$

By Solving Eq.(2-1),(2-2) and (2-3) Simultaneously,
Flow and Density Distribution can be Known Entirely.

To Grasp the Fundamental Behaviors of Density Flow,

Simplified Approximated Equation is Often Used.

Boussinesq Eq.

Simplified N.S Eq. Assuming Several Assumptions,

>Change of Density $\delta\rho$ due to Water Temp. etc is Smaller Compared with Standard Water Density $\left(\rho_0 \cong 1{,}000[kg/m^3]\right)$.

>Water Surface is Flat.

Derivation of Boussinesq Eq.

>Decompose Effects of Standard Density and those of Change of Density;

Decomposing Density; $\rho(t,\mathbf{x}) = \rho_0 + \delta\rho(t,\mathbf{x})$ (2-4)

- $\rho(t,\mathbf{x})$: Proper
- ρ_0: Standard Density (Constant)
- $\delta\rho(t,\mathbf{x})$: Difference from Standard Due to Water Temp., Salinity,….

Decomposing Pressure; $P(t,\mathbf{x}) = P_0 + \delta P(t,\mathbf{x})$ (2-5)

- $P(t,\mathbf{x})$: Proper
- P_0: Hydrostatic Pressure with Standard Density (ρ_0).
- $\delta P(t,\mathbf{x})$: Change of Pressure Due to $\delta\rho$.

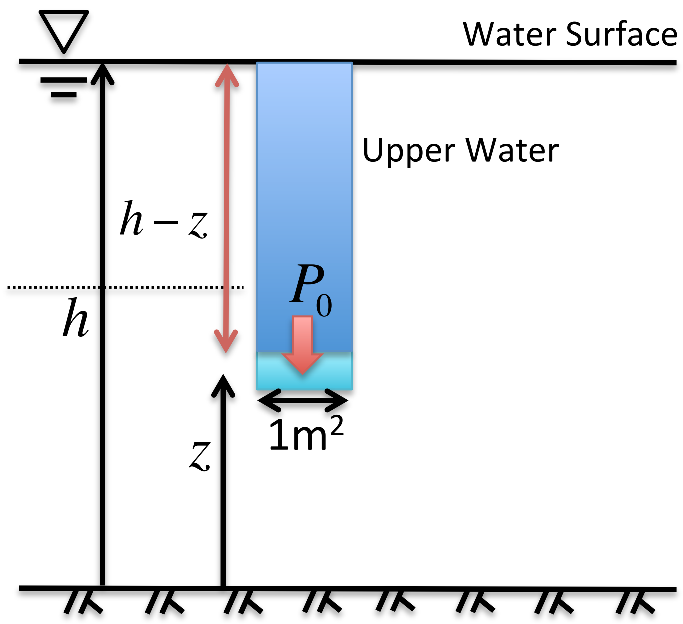
<u>Hydrostatic Pressure</u>

"Hydrostatic Pressure" = "Force per Unit Area ($1m^2$)
Caused by the Upper Water Pressing Down".

Vertical Position of Water Surface : $h[m]$

Volume of Upper Water : $(h-z)\times 1\times 1[m^3]$

Weight of Upper Water : $\rho_0 \times (h-z)$



<u>Hydrostatic Pressure at z :</u>

$$P_0 = g \times \text{"Weight of Upper Water"}$$

$$\therefore P_0 = g\rho_0 (h-z)$$

Spatial Gradient of Hydrostatic Pressure

Hydrostatic Pressure : $P_0 = g\rho_0 \left(h - z\right)$

By Assuming the Water Surface is Flat ($h = const.$),

$$\nabla P_0 = \begin{pmatrix} \dfrac{\partial P_0}{\partial x} \\ \dfrac{\partial P_0}{\partial y} \\ \dfrac{\partial P_0}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\rho_0 g \end{pmatrix} = \rho_0 \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} = \rho_0 \times \mathbf{F}_G \quad \text{(2-6)}$$

∇P_0 is equal to “Multiplying Standard Density and Gravity Force ($\rho_0 \times \mathbf{F}_G$).

Substituting Spatial Gradient (Eq.(2-6)) for N.S Equation (Eq.(2-1)).

N.S Eq. ; $$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{u} + \mathbf{F}_G \qquad (2\text{-}1)$$

・Substituting Decomposed Form $P = P_0 + \delta P$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla\left(P_0 + \delta P\right) + \rho\nu\Delta\mathbf{u} + \rho\mathbf{F}_G$$

・Expanding

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P_0 - \nabla\left(\delta P\right) + \rho\nu\Delta\mathbf{u} + \rho\mathbf{F}_G$$

・Substituting Eq.(2-6); $\nabla P_0 = \rho_0\mathbf{F}_G$

$$\rho\frac{D\mathbf{u}}{Dt} = -\rho_0\mathbf{F}_G - \nabla\left(\delta P\right) + \rho\nu\Delta\mathbf{u} + \rho\mathbf{F}_G$$

・Substituting Decomposed Form $\rho = \rho_0 + \delta\rho$

$$\rho\frac{D\mathbf{u}}{Dt} = -\rho_0\mathbf{F}_G - \nabla\left(\delta P\right) + \rho\nu\Delta\mathbf{u} + \rho_0\mathbf{F}_G + \delta\rho\mathbf{F}_G$$

・Eliminating $\rho_0\mathbf{F}_G$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla\left(\delta P\right) + \rho\nu\Delta\mathbf{u} + \delta\rho\mathbf{F}_G \qquad (2\text{-}7)$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla\left(\delta P\right) + \rho \nu \Delta \mathbf{u} + \delta\rho \mathbf{F}_G \qquad \text{(2-7)}$$

Dividing Eq.(2-7) by Standard Density ρ_0 ,

$$\frac{\rho}{\rho_0}\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla\left(\delta P\right) + \frac{\rho}{\rho_0}\nu \Delta \mathbf{u} + \frac{\delta\rho}{\rho_0}\mathbf{F}_G \qquad \text{(2-8)}$$

If We Assume "Change $\delta\rho$ " is Smaller Compared with "Standard ρ_0 ",

We can Approximate $\frac{\rho}{\rho_0} = \frac{\rho_0 + \delta\rho}{\rho_0} = 1 + \underline{\frac{\delta\rho}{\rho_0}} \cong 1$ (2-9)

negligibly small～0

By Substituting Eq.(2-9) for Eq.(2-8),

Expression of "Boussinesq Equation"

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla\left(\delta P\right) + \nu\Delta\mathbf{u} + \frac{\delta\rho}{\rho_0}\mathbf{F}_G$$

Writing in Non-vector Form,

$$\begin{cases} \dfrac{Du}{Dt} = -\dfrac{1}{\rho_0}\dfrac{\partial\left(\delta P\right)}{\partial x} + \nu\Delta u \\ \dfrac{Dv}{Dt} = -\dfrac{1}{\rho_0}\dfrac{\partial\left(\delta P\right)}{\partial y} + \nu\Delta v \\ \dfrac{Dw}{Dt} = -\dfrac{1}{\rho_0}\dfrac{\partial\left(\delta P\right)}{\partial z} + \nu\Delta u - g' \end{cases} \qquad (2\text{-}10)$$

Relative Buoyancy : $g' \equiv \dfrac{\delta\rho}{\rho_0} g = \dfrac{\rho - \rho_0}{\rho_0} g$ (2-11)

By Changing the Symbol to represent Pressure; $\delta P \rightarrow P$

Boussinesq Eq.

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \nu\Delta u$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \nu\Delta v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \nu\Delta u - g'$$

N.S. Eq. for Constant Density

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \nu\Delta u$$

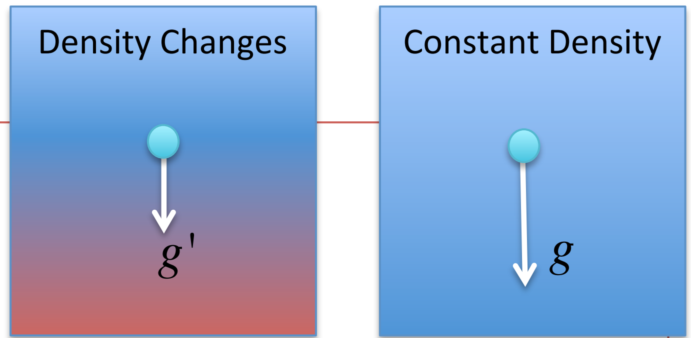
$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \nu\Delta v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \nu\Delta u - g$$

Difference Between Boussinesq Eq. and N.S. Eq. for Constant Density; Only the Gravity Term.

In Flow with Spatial Change of Density,

Relative Buoyancy g' is Virtually Affected as Vertical Gravity Force Instead of g .



Physical Meaning of Relative Buoyancy g'.

Vertical Force $-g'$ Depends on Difference of Density; $-g' = -\dfrac{\rho - \rho_0}{\rho_0} g$

Small Water Element (Density ρ).

○ Density ρ is Heavier than ρ_0 ($\rho > \rho_0$)

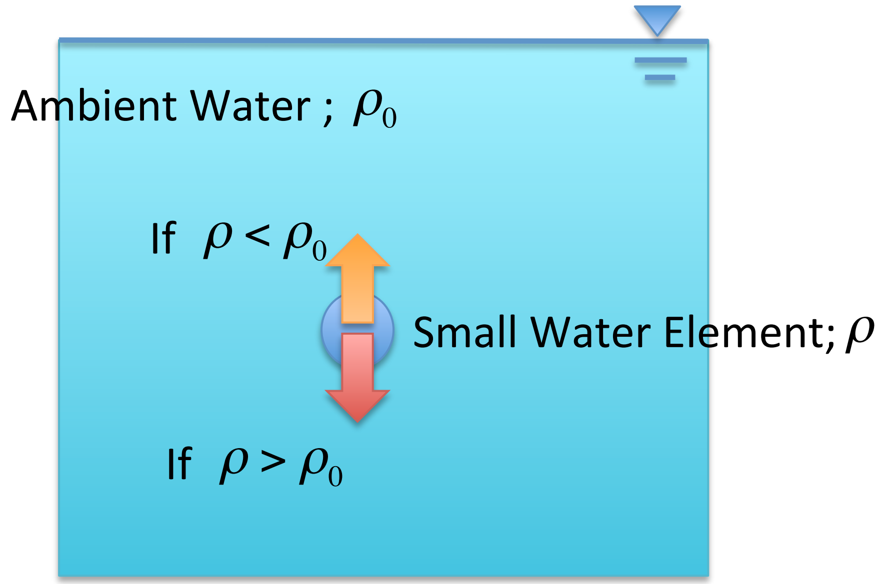
$$-g' = -\frac{\rho - \rho_0}{\rho_0} g < 0$$

Downward Force Affects on Element

○ Density ρ is Lighter than ρ_0 ($\rho < \rho_0$)

$$-g' = -\frac{\rho - \rho_0}{\rho_0} g > 0$$

Upward Force Affects on Element



→ g' Represents Effects of Buoyancy That We See in Daily Life.

Summary of §2.2

>When We can Assume that,
Density Change is Smaller than Standard Density,
and
Water Surface dose not Fluctuate (Flat Surface),
Water Flow with Spatial Density Change can be Represented by Boussinesq Eq..

>In the Water Flow with Spatial Density Change,
"Relative Buoyancy g' " is Affecting on the Water Bulks instead of g .

>"Relative Buoyancy g' " Means Buoyancy Force Depending on the Difference of Density.
If Water Element is Heavier than Ambient Water, Downward Force Affects,
Otherwise, Upward Force Affects.