§2.4 Feature of Dynamics of Density Stratification – Oscillation & Stability of Stratification

Under a Stationary Flow Field $(\mathbf{u}=0)$.

>Horizontal Stratification.

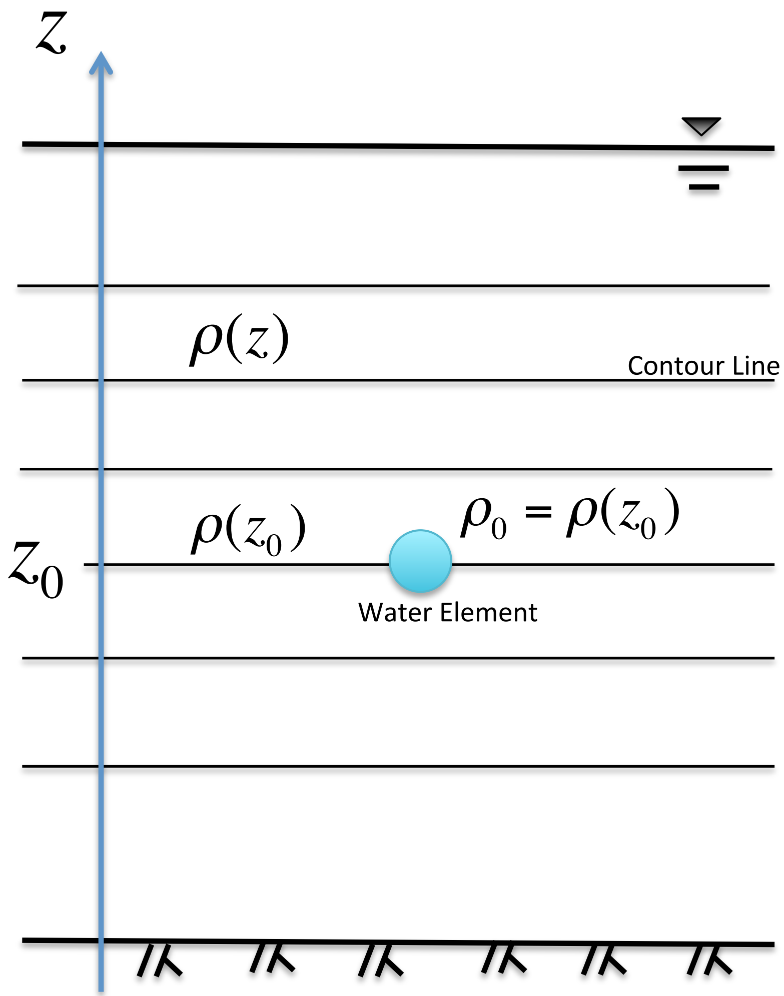
>Density is a function of “z”;

>Investigate Movement
of Small Water Element.

At Initial

Located at z_0

Density is $\rho_0 = \rho(z_0)$



> Assuming Element was Shifted Upward
Due to Something (Perturbation).

>Suppose Displacement
from Initial Position; η

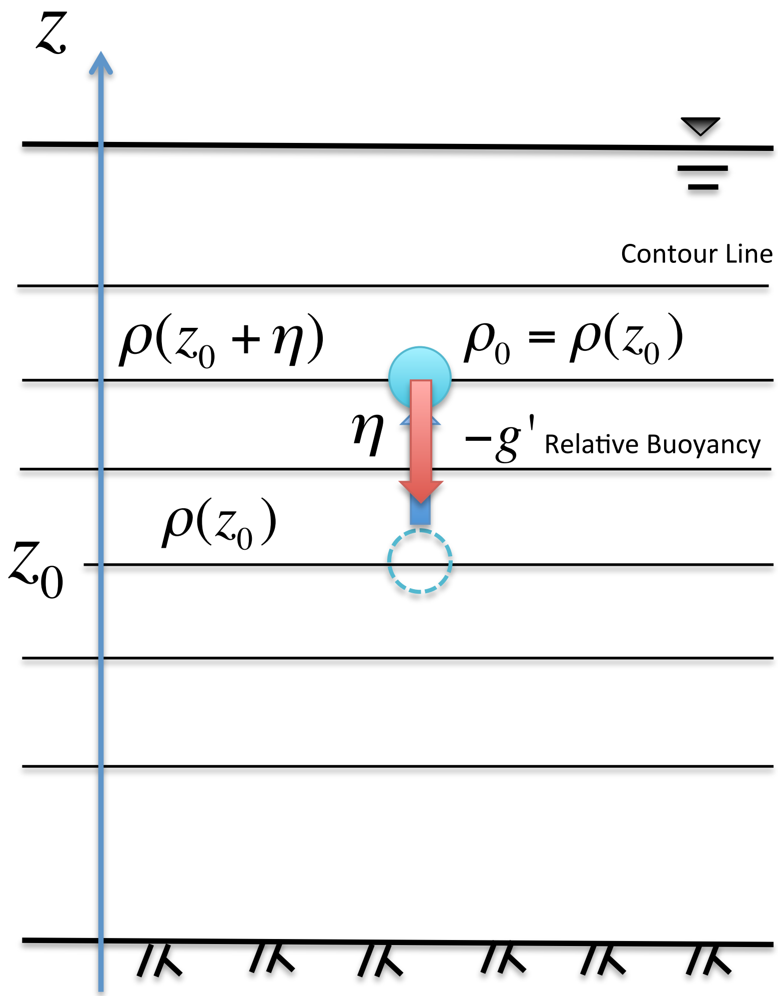
> Vertical Position of Element; $z_0 + \eta$

> After Shifting Upward,
Density of Ambient Water Changes

$$\rho(z_0) \Rightarrow \rho(z_0 + \eta)$$

> Density of Element is Kept $\rho_0 = \rho(z_0)$.

> Relative Buoyancy $-g'$ Affects on Element.



$$-g' = -\frac{"Density\_Difference"}{"Ambient\_Density"} g = -\frac{\rho_0 - \rho(z_0 + \eta)}{\rho_0} g$$

> Using Taylor Expansion,

$$\rho(z_0+\eta)=\rho(z_0)+\eta\frac{\partial\rho(z_0)}{\partial z}+\frac{\eta^2}{2!}\frac{\partial^2\rho(z_0)}{\partial z^2}+\dots$$

> Assuming Displacement is Small; $\eta << 1$

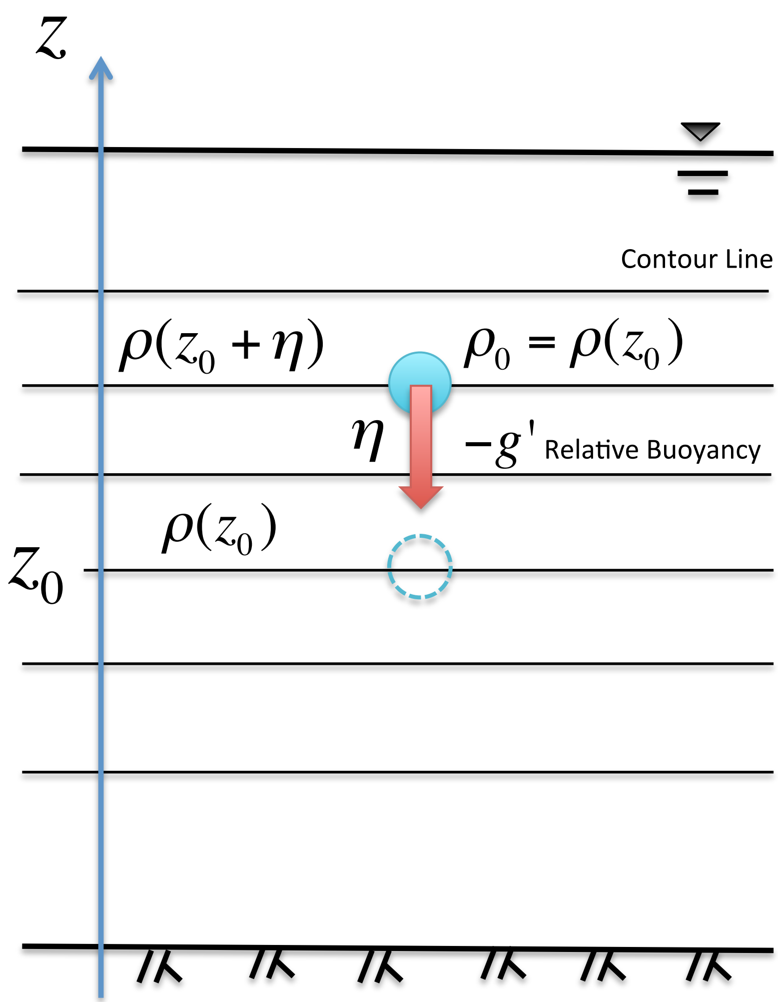
$$\rho(z_0+\eta)\cong\rho(z_0)+\eta\frac{\partial\rho(z_0)}{\partial z}$$

> Relative Buoyancy $-g'$ is rewritten as a function of η;

$$-g'=-\frac{\rho_0-\rho(z_0+\eta)}{\rho_0}g$$

$$-g'=-\frac{\rho_0-\left\{\rho(z_0)+\eta\frac{\partial\rho(z_0)}{\partial z}\right\}}{\rho_0}g$$

$$\therefore -g'=-\left(-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right)\times\eta$$



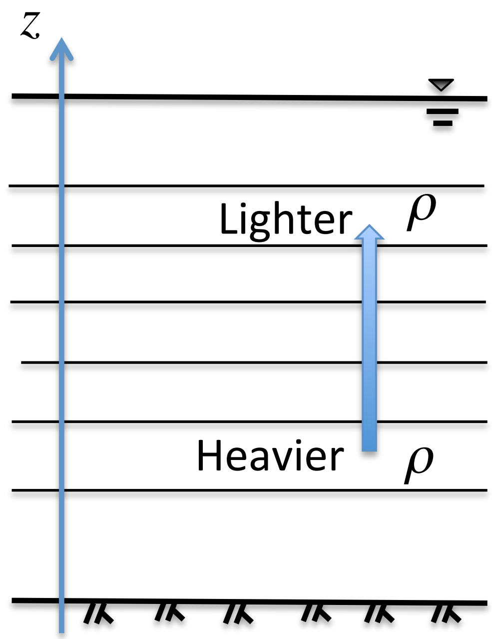
> Equation of Element's Motion

$$\frac{d^2\eta}{dt^2} = -g'$$

$$\therefore \frac{d^2\eta}{dt^2} = -\left(-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right)\times\eta$$

> In the Case of $\frac{\partial\rho}{\partial z} < 0$ ⇒ Upper Water is Lighter

$$\because \left(-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right) > 0$$



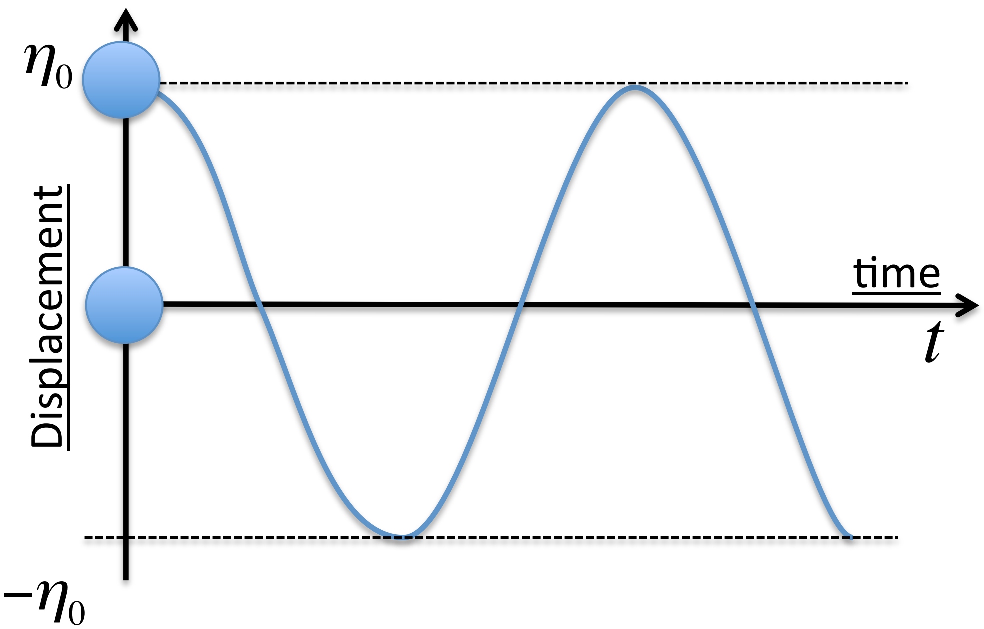
<u>Theoretical Solution;</u> $\eta = \eta_0 \cos\left\{\sqrt{-\frac{g}{\rho}\frac{\partial\rho}{\partial z}}\times t\right\}$

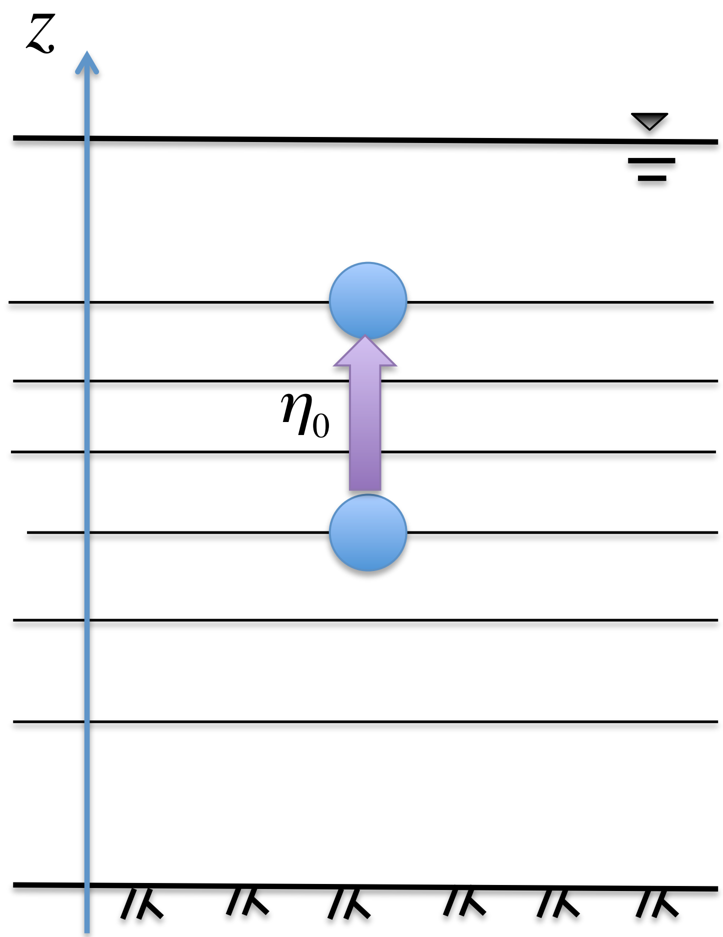
η_0 :Initial Displacement at $t = 0$.

> In the Case of $\frac{\partial \rho}{\partial z} < 0$ ➡ Upper Water is Lighter

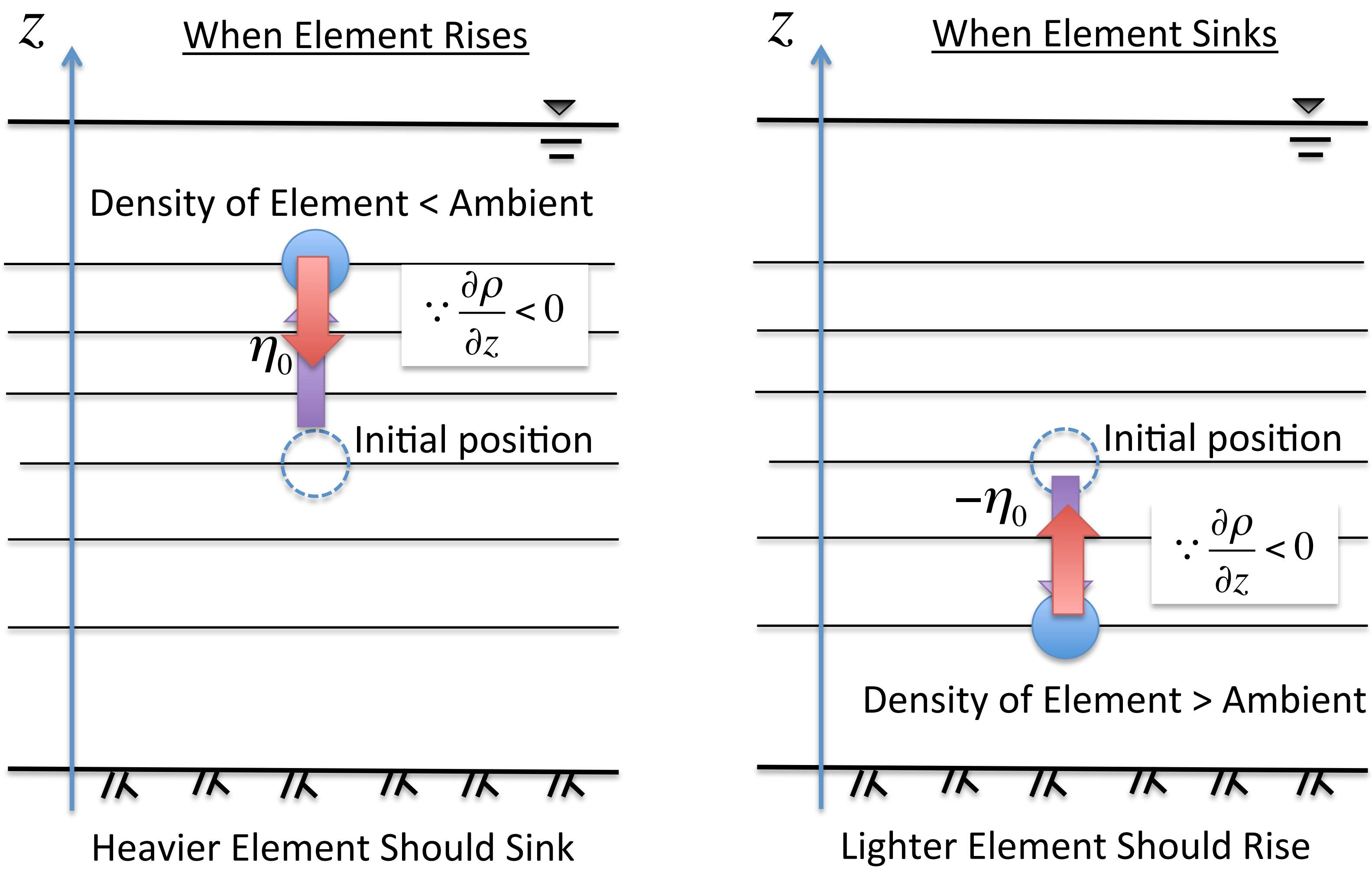
Theoretical Solution; $\eta = \eta_0 \cos\left\{\sqrt{-\frac{g}{\rho}\frac{\partial \rho}{\partial z}} \times t\right\}$ ➡ Element Oscillates with Constant Amplitude η_0 .

η_0 :Initial Displacement at $t = 0$.





Oscillation Behavior in the Case of $\frac{\partial \rho}{\partial z} < 0$



Element is always Affected by the Force to Put Back the Original Position

Oscillation in the Case of $\frac{\partial \rho}{\partial z} < 0$: Brunt-Vaisala Oscillation

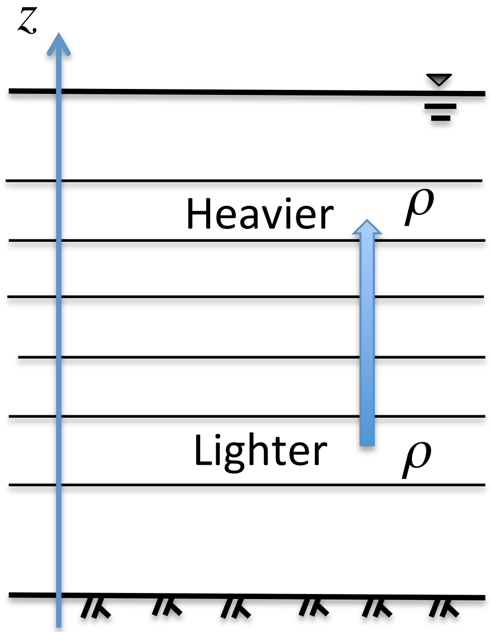
$N \equiv \sqrt{-\frac{g}{\rho}\frac{\partial \rho}{\partial z}}$: Brunt-Vaisala Frequency

> Equation of Element's Motion

$$\therefore \frac{d^2\eta}{dt^2} = -\left(-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right)\times\eta$$

> In the Case of $\frac{\partial\rho}{\partial z} > 0$ ➡ Upper Water is Heavier

$$\because \left(-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right) < 0$$

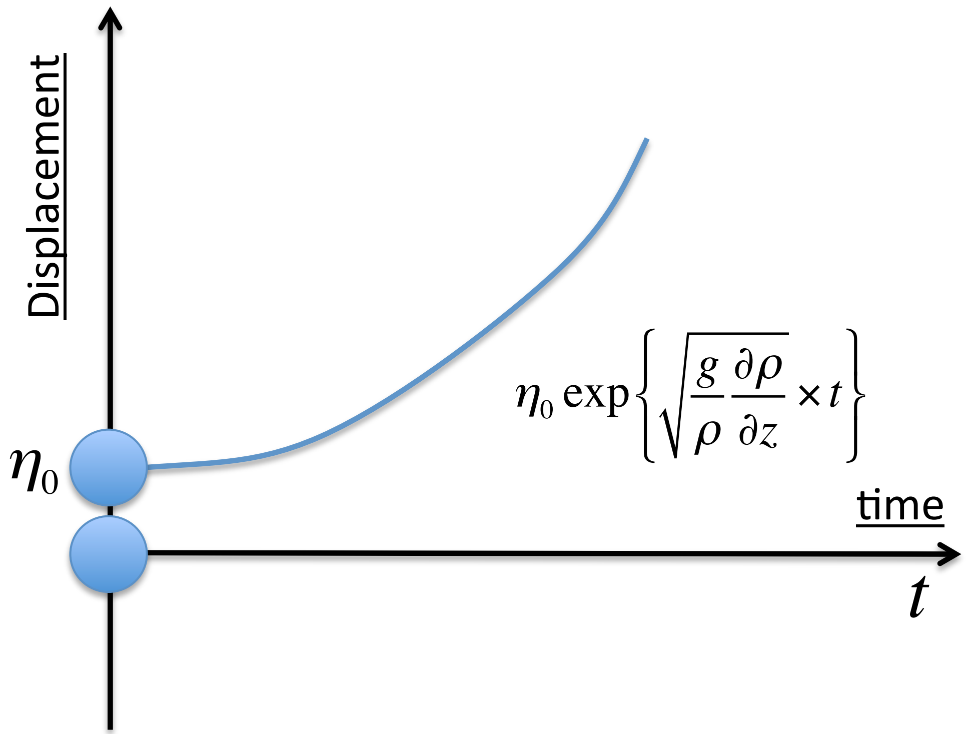


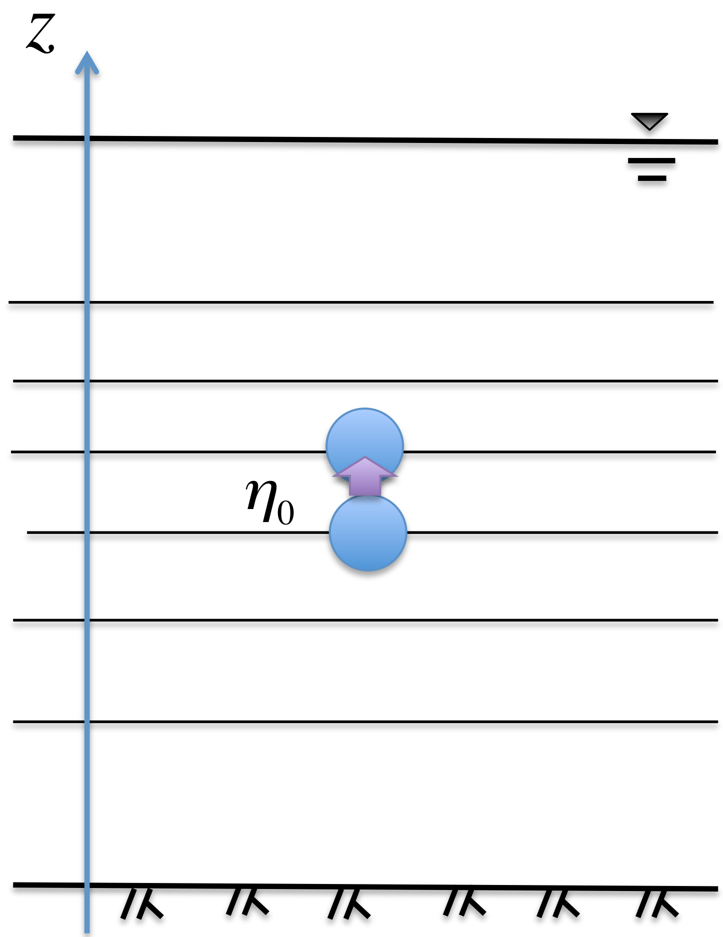
<u>Theoretical Solution;</u> $\eta = \eta_0 \exp\left\{\sqrt{\frac{g}{\rho}\frac{\partial\rho}{\partial z}}\times t\right\}$

η_0 :Initial Displacement at $t = 0$.

> In the Case of $\frac{\partial \rho}{\partial z} > 0$ → Upper Water is Heavier

Theoretical Solution; $\eta = \eta_0 \exp\left\{ \sqrt{\frac{g}{\rho}\frac{\partial \rho}{\partial z}} \times t \right\}$ η_0 :Initial Displacement at $t = 0$.





Even if Initial Displacement is Very Small,

Displacement Grows Exponentially and Endlessly.

Movement of Element Depends on Vertical Gradient of Density

$$\frac{\partial \rho}{\partial z} < 0$$: Element Oscillates Stably

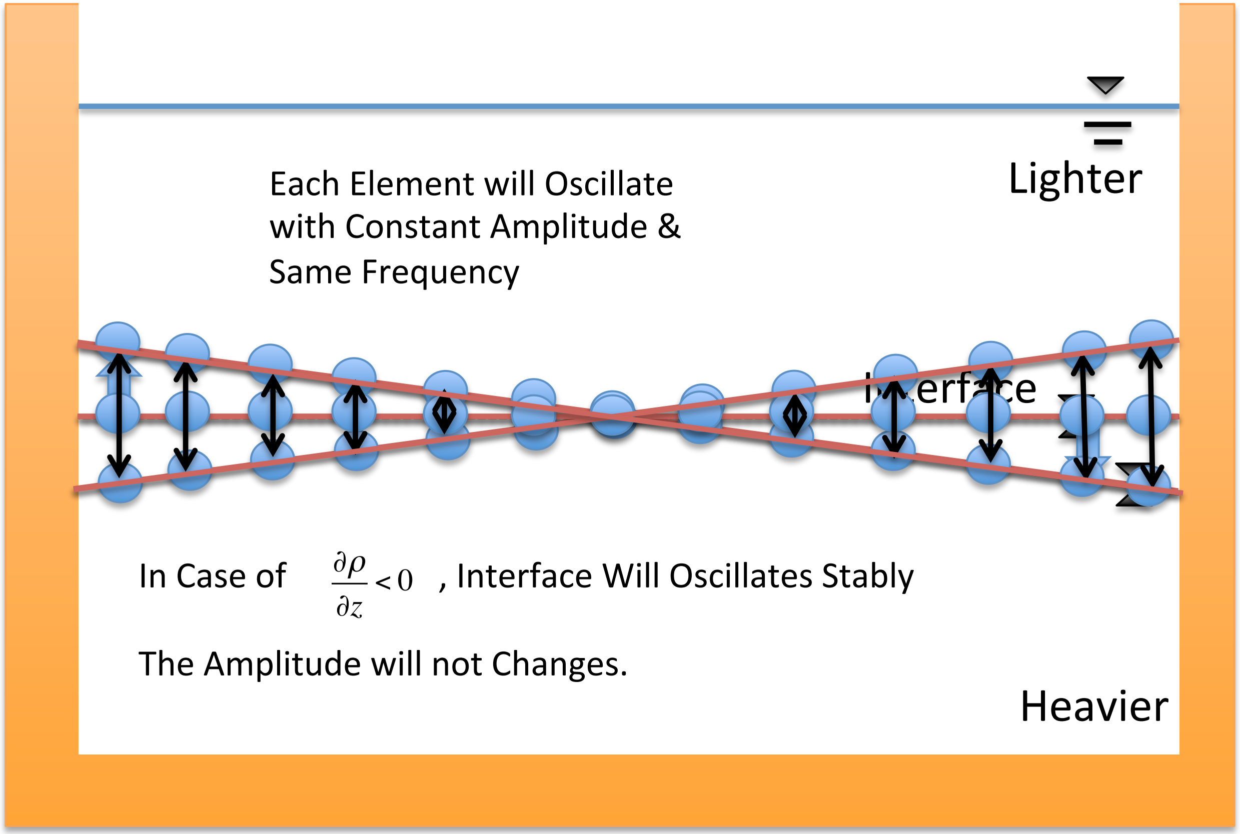
$$\frac{\partial \rho}{\partial z} > 0$$: Element Rises/Sinks Endlessly

Evaluate Feature of Movement of Interface of Stratification Based on Above Facts

>In Case of $\frac{\partial \rho}{\partial z} < 0$ ➡ Element Oscillates Stably

>Suppose Interface Inclines Due to Something (eg, Wind Set-Up),

>How Interface will Move since Then can be Evaluated by Considering Elements Located on Interface.

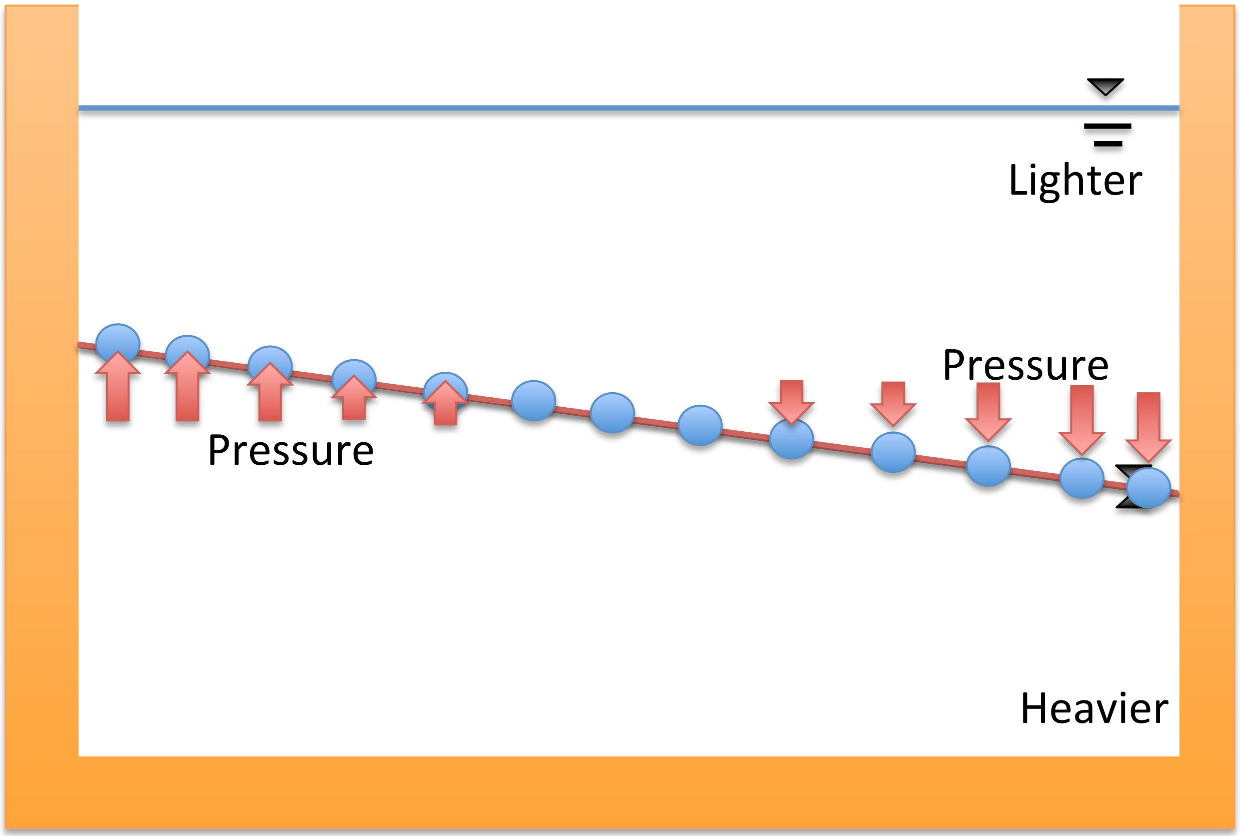


※ Oscillation of Interface is Named as "Internal Wave" or "Seiche"

Although Case of “Internal Wave”/ “Seiche” is Explained by Element Movement,

Frequency of “internal Wave”/ “Seiche” Generally Differs from Brunt-Vaisala Frequency

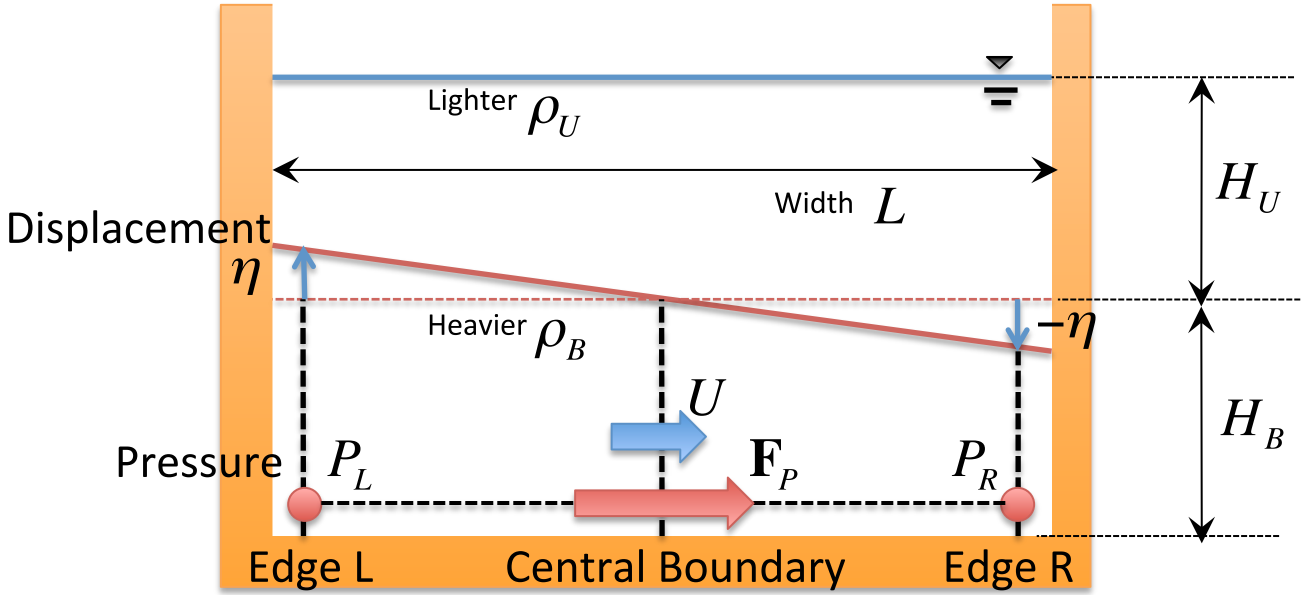
Because the Elements are Affected by Pressure Force.



Evaluation of Frequency of "internal Wave" Taking Account of Pressure.

>Assuming Displacement ; η

>Suppose the Horizontal Velocity at Central Boundary in Bottom Layer is U.



>Pressure at Edge L ;

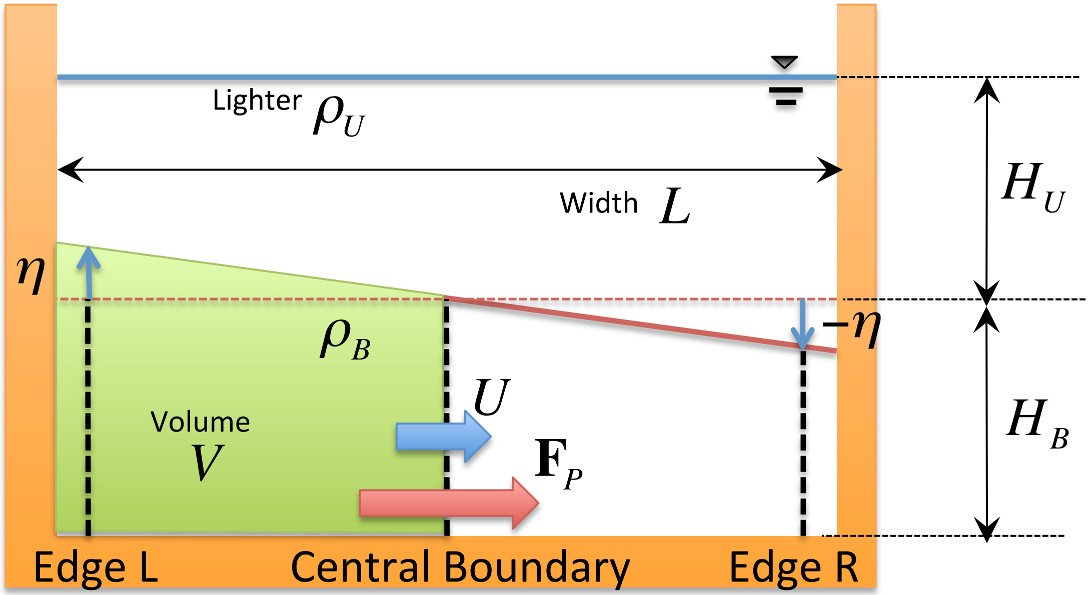
$$P_L = g(H_U - \eta)\rho_U + g\left(H_B + \eta\right)\rho_B$$

>Pressure at Edge R ;

$$P_R = g(H_U + \eta)\rho_U + g\left(H_B - \eta\right)\rho_B$$

Pressure Force at Central Boundary;

$$\mathbf{F}_P = -\frac{1}{\rho_B}\frac{\partial P}{\partial x} \cong -\frac{1}{\rho_B}\frac{P_R - P_L}{L} = \frac{2g\eta}{\rho_B L}\left(\rho_U - \rho_B\right)$$

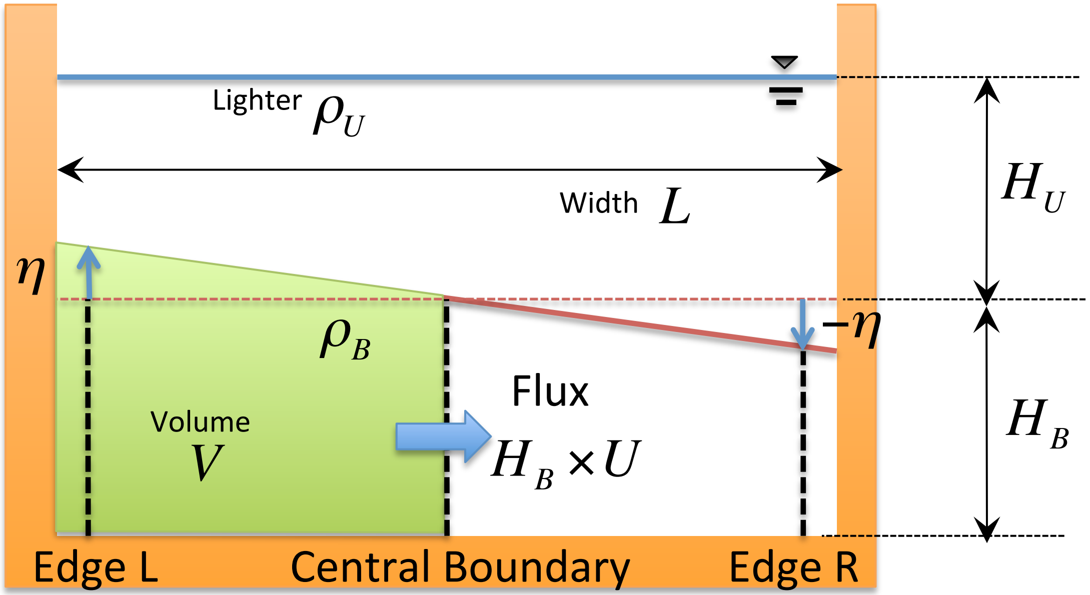


<u>U is Accelerated by F<u>P</u>:</u> $\frac{dU}{dt} = \mathbf{F}_P$

$$\therefore \frac{dU}{dt} = -\frac{2g\eta}{\rho_B L}(\rho_U - \rho_B)$$

<u>Flux Across the Central Boundary is Estimated by Multiplying with Thickness:</u> $U \times H_B$

<u>Volume of Right hand Side (Colored in Green) :</u> $V = \left(\frac{L}{2} \times H_B + \frac{\eta}{2} \times \frac{L}{2}\right) = \frac{LH_B}{2} + \frac{\eta L}{4}$

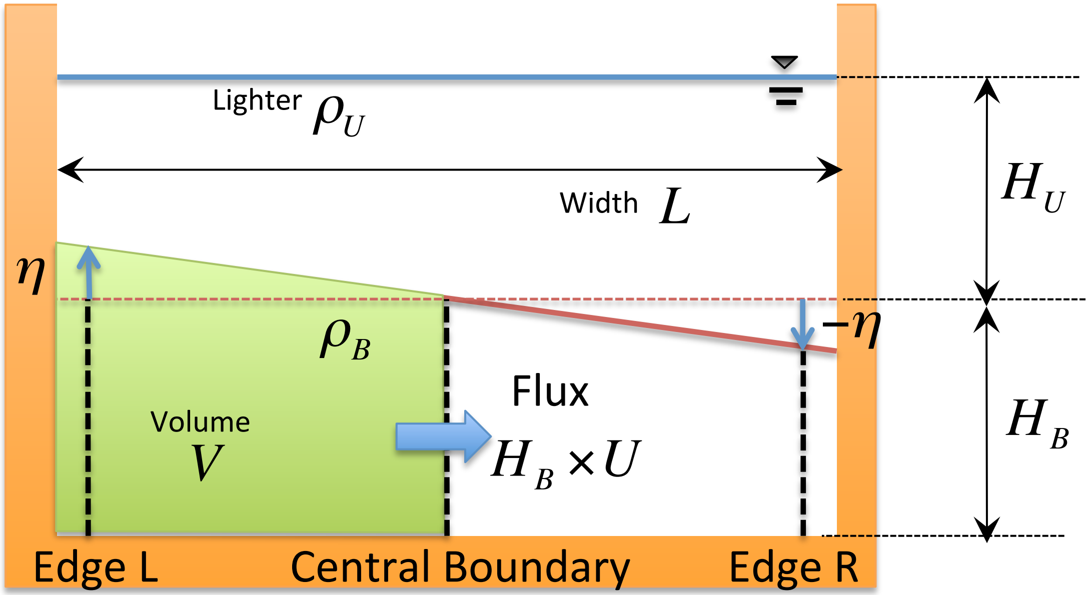


Volume Conservation : "Change of Volume" is "Flux at Central Boundary"

$$\frac{dV}{dt} = -UH_B$$

$$\frac{d\left(\frac{LH_B}{2} + \frac{L\eta}{4}\right)}{dt} = -UH_B$$

$$\therefore \frac{d\eta}{dt} = -\frac{4H_B}{L}U$$



Taking Derivative in Time :

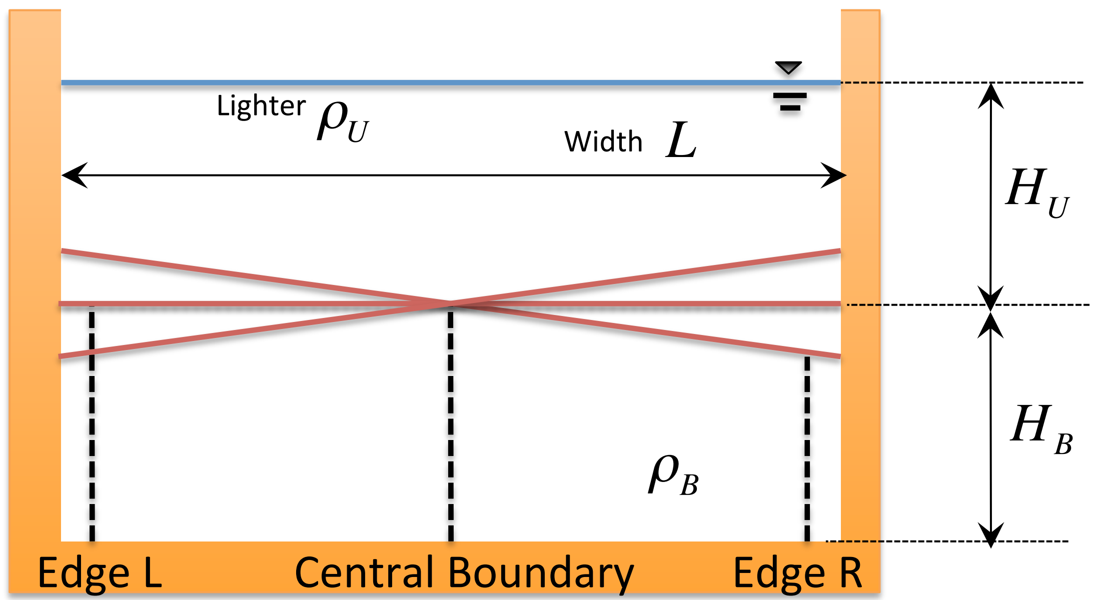
$$\frac{d^2\eta}{dt^2} = -\frac{4H_B}{L}\frac{dU}{dt}$$

Substituting

$$\frac{dU}{dt} = -\frac{2g\eta}{\rho_B L}(\rho_U - \rho_B)$$

$$\frac{d^2\eta}{dt^2} = -\frac{4H_B}{L} \times \frac{2g\eta}{\rho_B L}(\rho_B - \rho_U)$$

$$\therefore \frac{d^2\eta}{dt^2} = -N_{Seiche}^2 \eta \qquad N_{Seiche} \equiv \frac{2}{L}\sqrt{\frac{2gH_B(\rho_B - \rho_U)}{\rho_B}}$$



Theoretical Solution of $\dfrac{d^2\eta}{dt^2} = -N_{Seiche}^2\eta$ ⇨ $\eta = \eta_0 \cos(N_{Seiche} t)$

Interface Oscillate with Frequency N_{Seiche} .

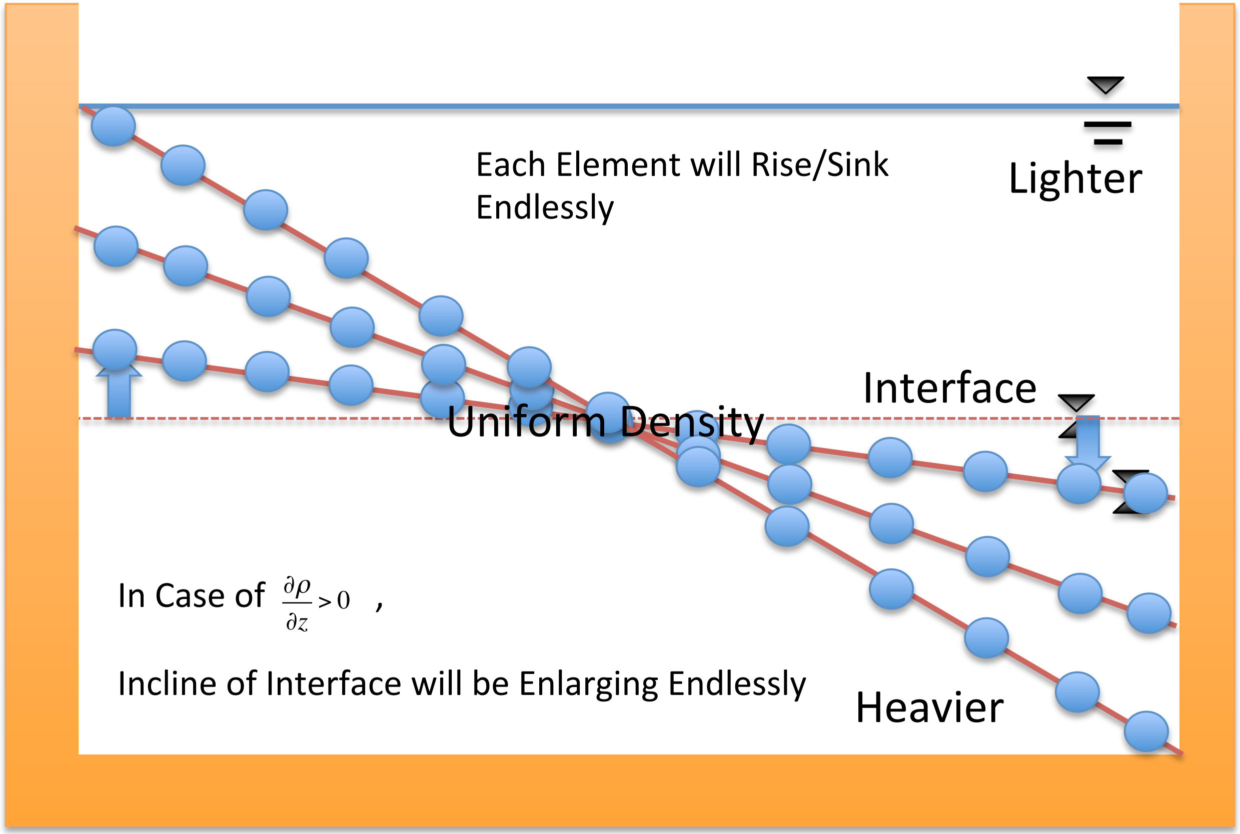
More Precise Frequency of "internal Wave"

Derived from Potential Theorem; $N_{Seiche} \equiv \dfrac{1}{L}\sqrt{\dfrac{gH_B(\rho_B - \rho_U)}{\rho_B}} = \dfrac{1}{L}\sqrt{H_B \times g'}$

Relative Buoyancy $g' = \dfrac{\rho_B - \rho_U}{\rho_B}$

>In Case of $\frac{\partial \rho}{\partial z} > 0$ ➡ Element Rises/Sinks Endlessly

>Even if the Initial Displacement is very Small,



※Finally, Interface (Stratification) will be Broken & Density Becomes Uniform Due to Strong Mixing.

Stability & Dynamics of Stratification

In Case of $\frac{\partial \rho}{\partial z} < 0$ (Upper Layer is Lighter than Bottom)

> Interface Oscillates with $N_{Seiche} \cong \frac{1}{L}\sqrt{H_B \times g'}$

> Amplitude will not Grows

> Stratification is Kept Stably

→ Named as "Stable (Normal) Stratification"

In Case of $\frac{\partial \rho}{\partial z} > 0$ (Upper Layer is Heavier than Bottom)

> Displacement of Interface will Grow Exponentially & Endlessly.

> Stratification can be Kept Stably & Broken Immediately.

> Density will Becomes Uniform due to Immediate Mixing.

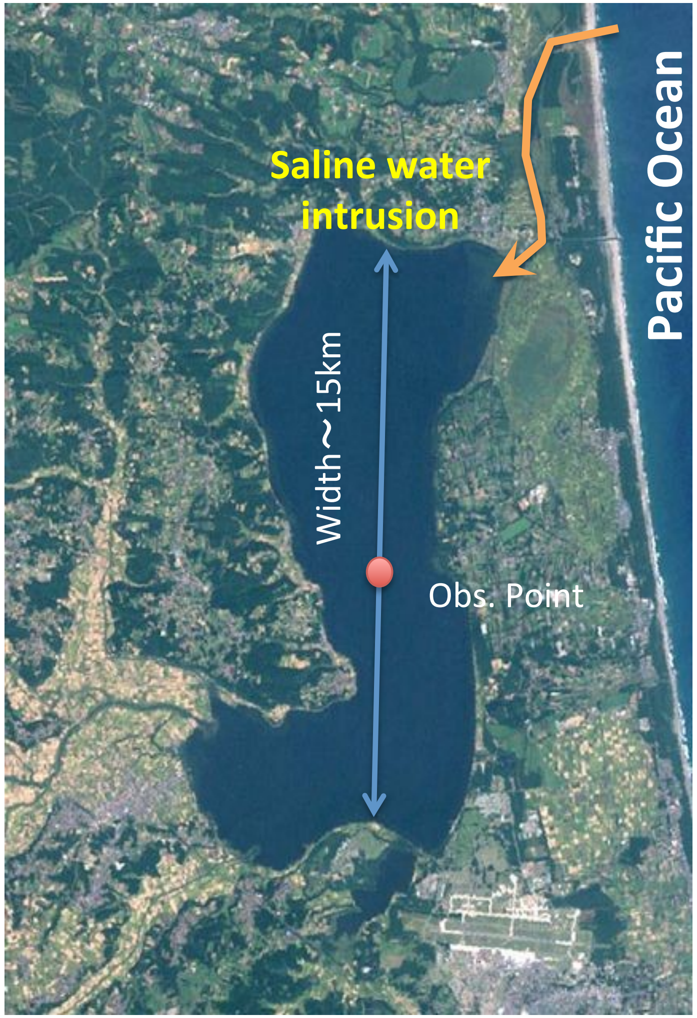
→ Named as "Unstable (Inverse) Stratification"

Example in Actual Water Environment

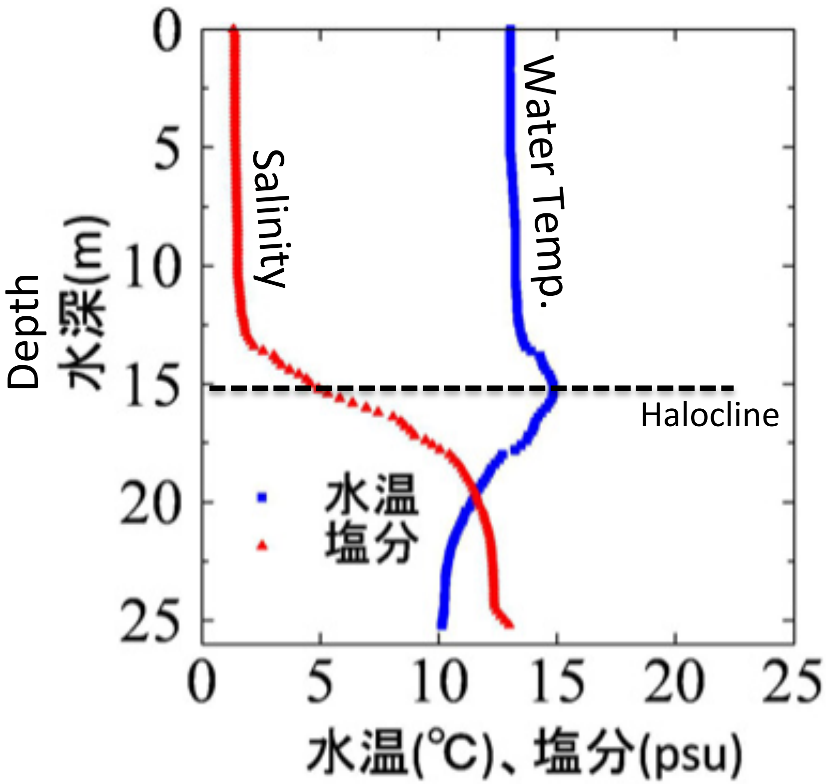
> Ogawara Lake (小川原湖), Aomori Prefecture, Japan

> Lake Neighbors on Pacific Ocean & Sea Water Intrudes into Lake

> Lake is Brackish & Saline Water Layer Exists (Halocline Locates around 15m Depth).

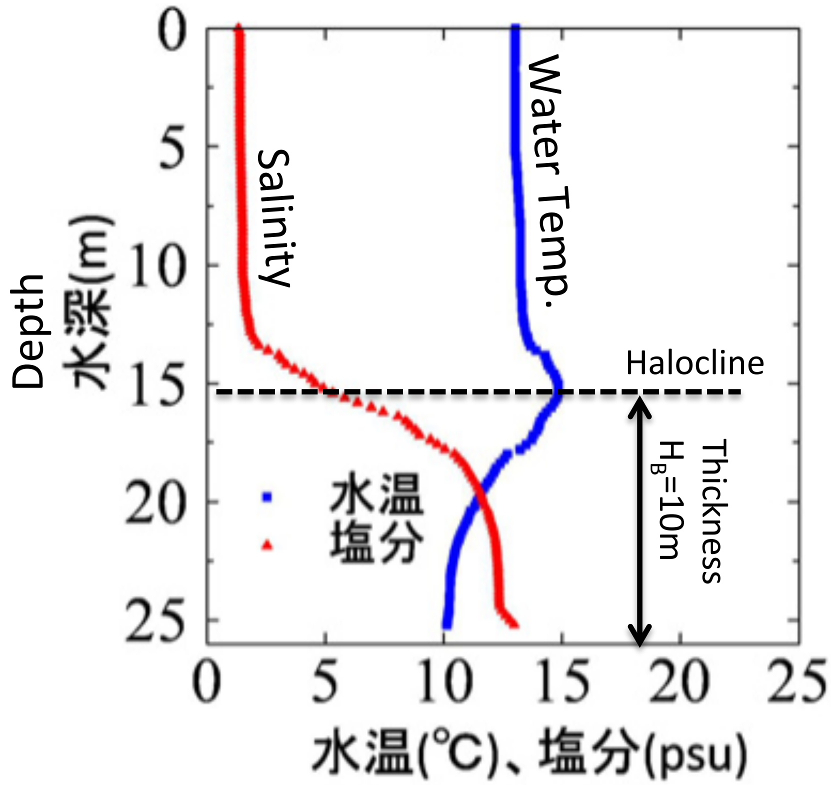


Lake Ogawara
(Brackish water Lake)



the salinity profile at the lake center (Winter)

> Stable (Normal) Stratification Condition.



Salinity

Upper Fresh Water Layer : 3 [‰]

Bottom Saline Water Layer : 12 [‰]

Density

Upper Layer : $\rho_U \cong 1{,}002[kg/m^3]$

Bottom Layer : $\rho_B \cong 1{,}009[kg/m^3]$

Relative Buoyancy

$$g' = \frac{\rho_B - \rho_U}{\rho_B} g$$

$$= \frac{1009 - 1002}{1009} \times 9.8 = 0.068[m/s^2]$$

Expected Frequency of "Internal Wave"

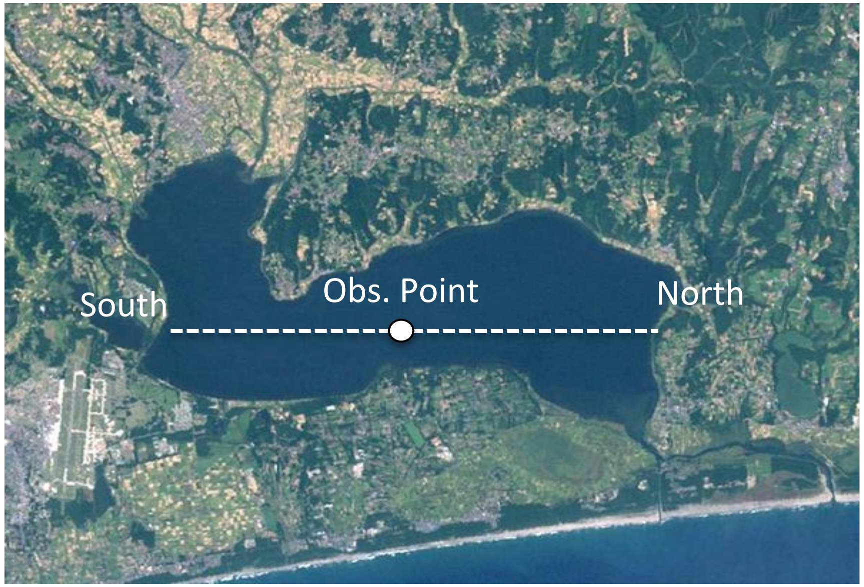
$$N_{Seiche} = \frac{1}{L}\sqrt{H_B \times g'} = \frac{1}{15[km]}\sqrt{10[m] \times 0.068[m/s^2]} = 0.000055[1/\sec]$$

Expected Period of "Internal Wave" $T_{Seiche} = \frac{2\pi}{N_{Seciche}} = 114{,}292[\sec] \cong 32[hour]$

Obs. Results

>Obs. Is Conducted at Central Lake Using Acoustic Doppler Current Profiler(ADCP).

>Time Series of Vertical Profile of Velocity is Observed.



ADCP

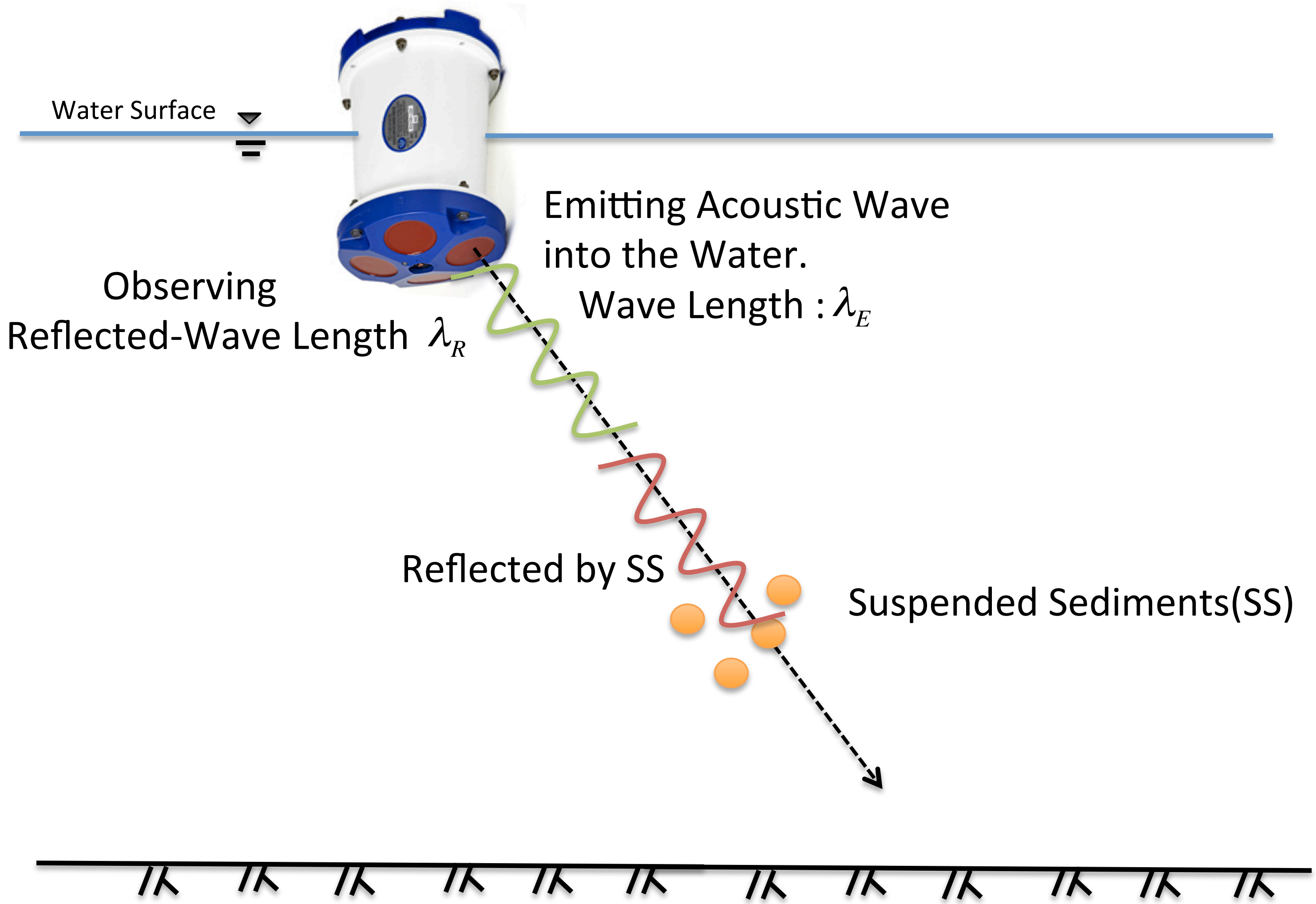
Emitting Acoustic Wave into the Water.

Emitting Device is Soaked in Water.

Acoustic Doppler Current Profiler (ADCP)

> If SS Stops (Flow Velocity u=0), "Emitted Wave Length λ_E "="Reflected Wave Length λ_R "

> If There are Flows (u≠0), "Reflected Wave Length" Changes by "Doppler Shift".

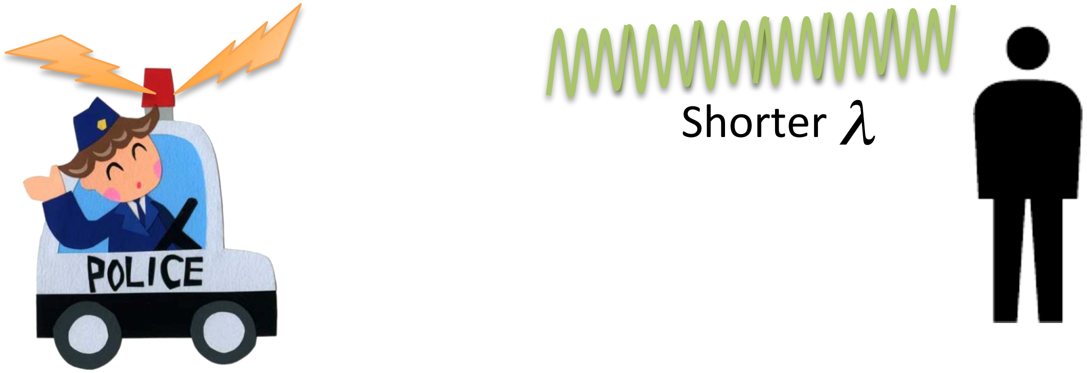


Doppler Shift

> You Experience "Doppler Shift" in Daily Life.

> A Car Sounding a Siren,

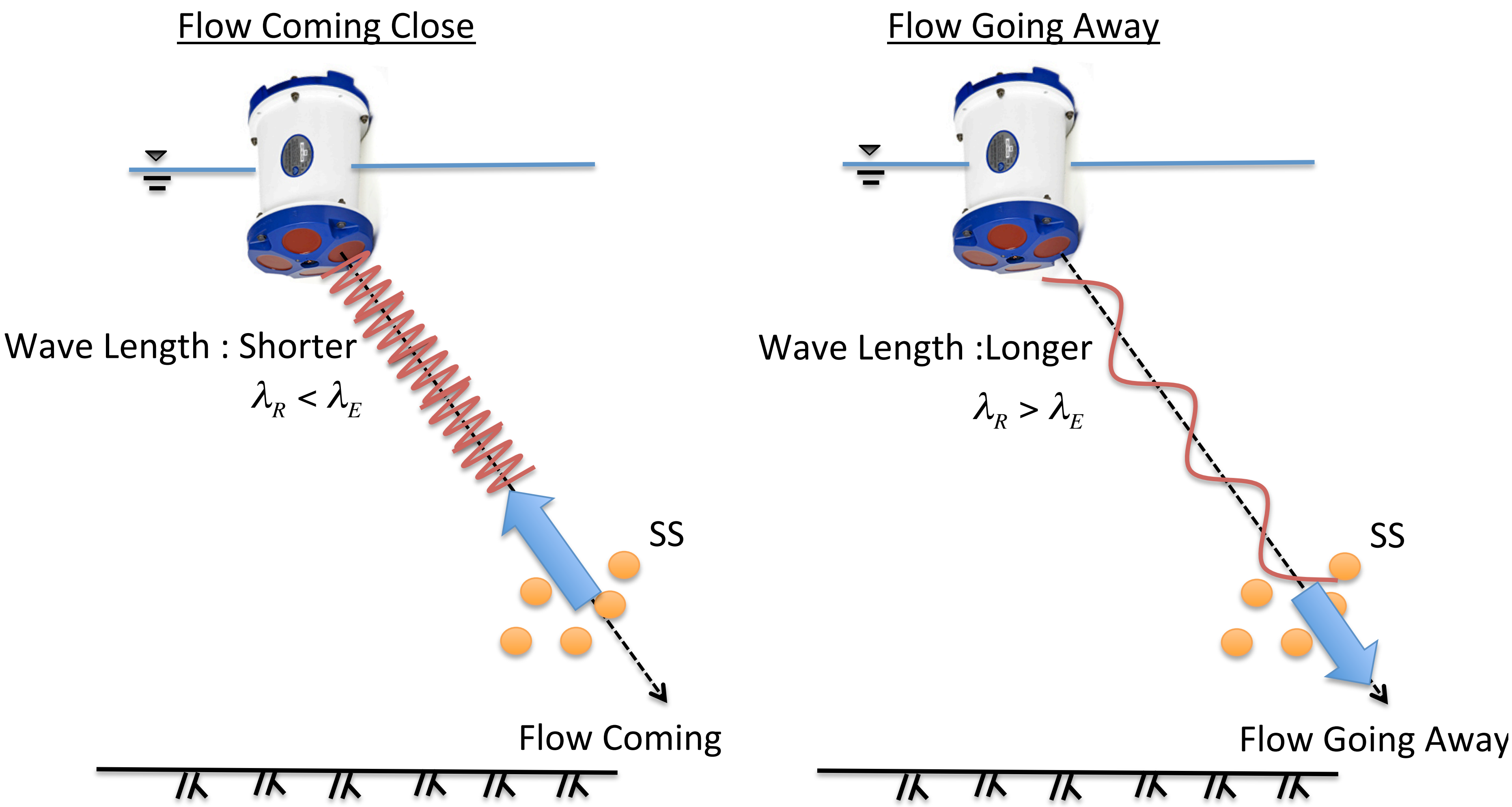
When a Car is Coming → Sound Changes High Tone (Shorter Wave Length)



When a Car is Going Away → Sound Changes Low Tone (Longer Wave Length)

Acoustic Doppler Current Profiler (ADCP)

> If There are Flows (u≠0) & SS Moves, "Reflected Wave Length" Changes.

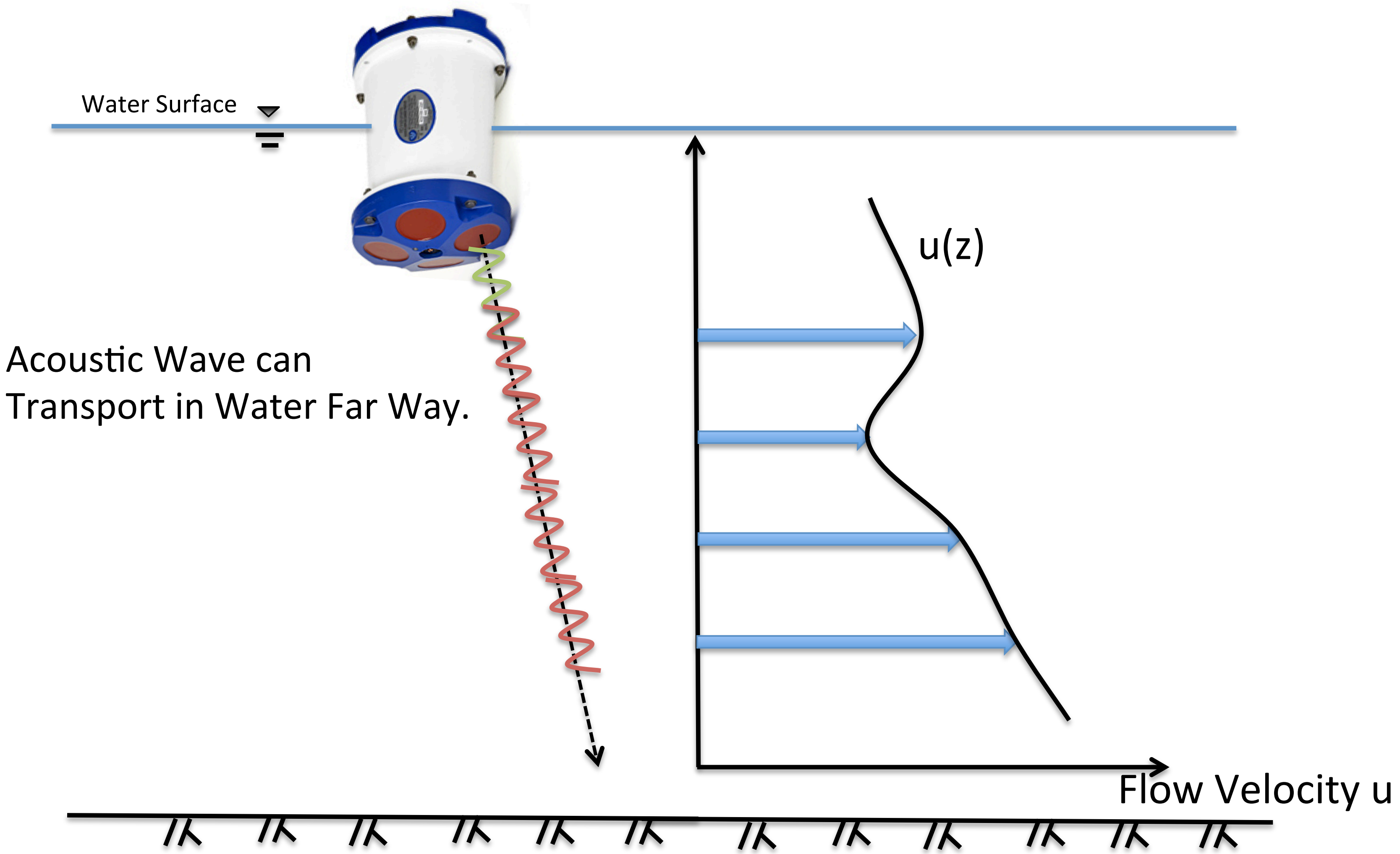


※ By Detective Change of Wave Length, Flow Velocity can be Observed.

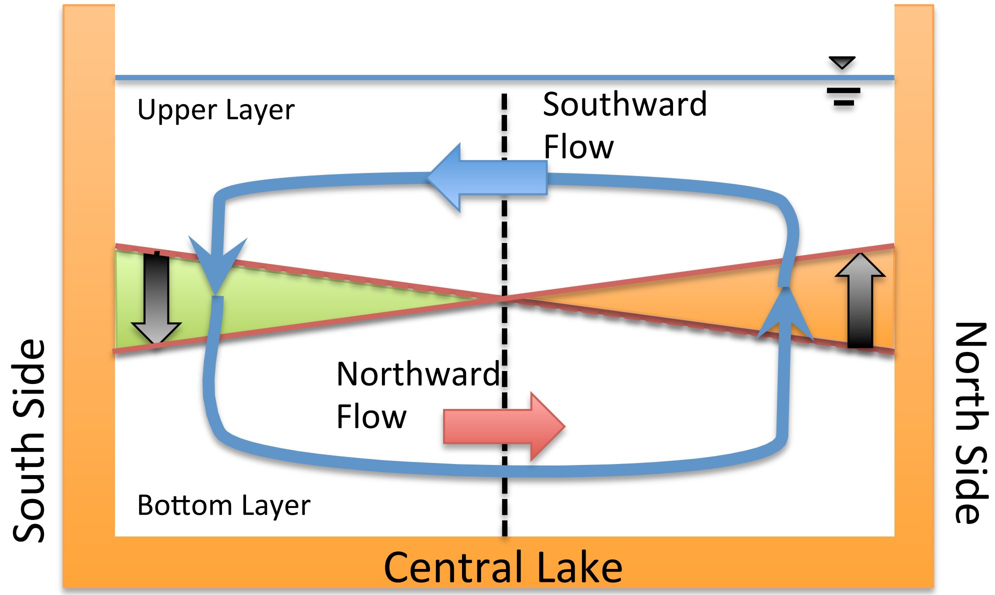
Acoustic Doppler Current Profiler (ADCP)

> Acoustic Wave can Transport in Water Far Way, Although a Part of it is Reflected.

> By Detecting Reflected Wave from Each Depth, Vertical Profile of Velocity is Measured.



Expected Velocity Vertical Profile If "Internal Wave" Exists



When Interface at South Side Sinks,

in Bottom Layer Green Water Must be Moved to Orange Area.

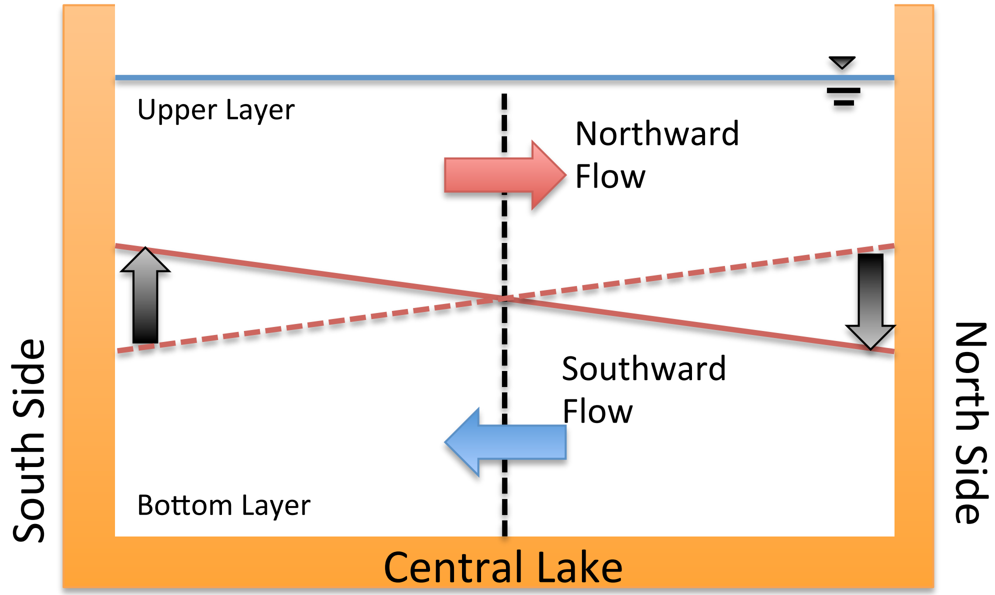
→ Northward Flow is Generated at Central Lake.

in Upper Layer Orange Water Must be Moved to Green Area.

→ Southward Flow is Generated at Central Lake.

※ When "Internal Wave" Exists, Horizontal Flow is Generated
& Flow Direction of Each Layer is Opposite to Each Other.

Expected Velocity Vertical Profile If “Internal Wave” Exists



After Time Passed, When Interface at South Side Rises Up,

In Same Manner,

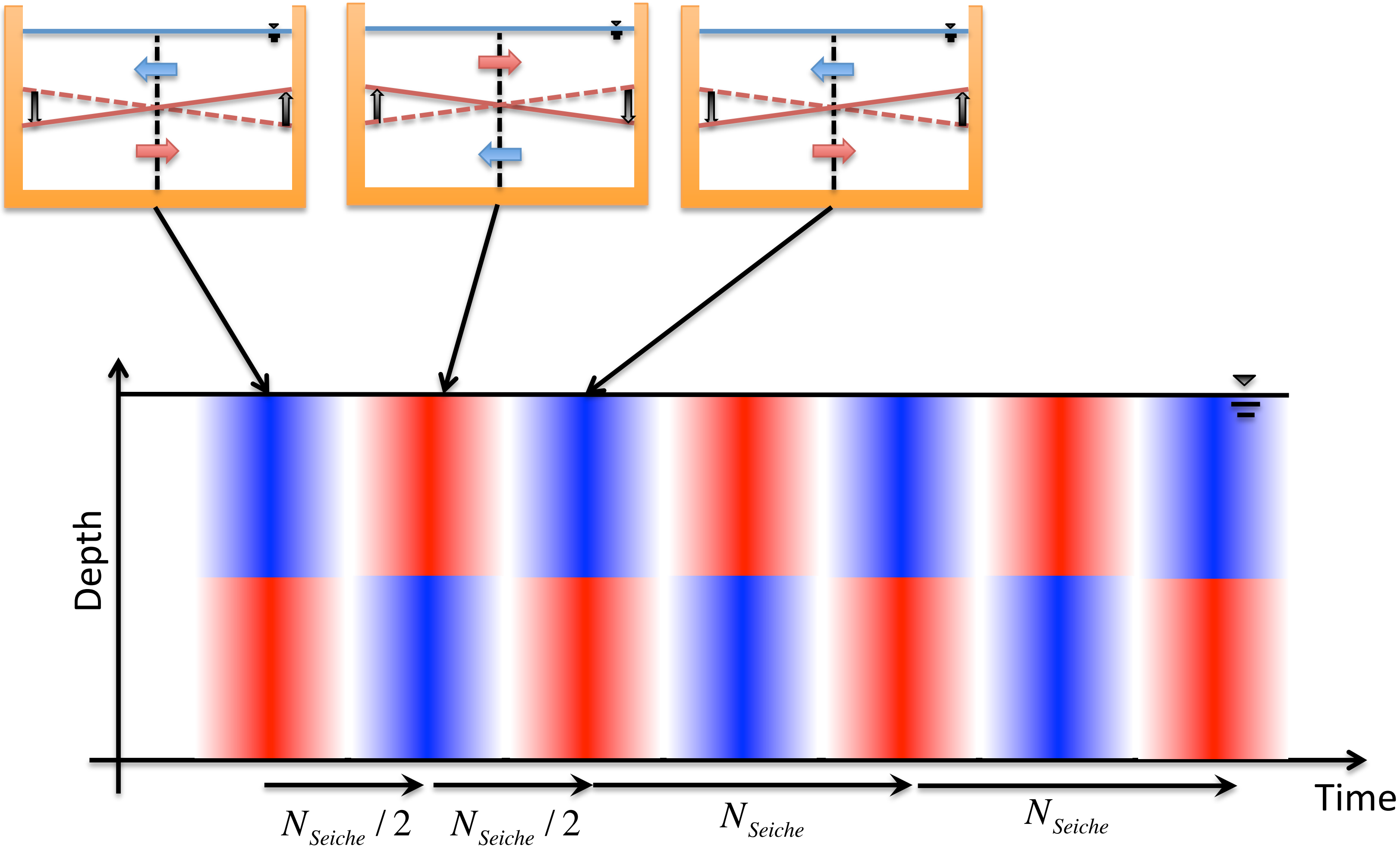
in Bottom Layer → Southward Flow is Generated at Central Lake.

in Upper Layer → Northward Flow is Generated at Central Lake.

※ Flow Direction Changes Periodically with Same Time Period of “Internal Wave”.

Expected Velocity Vertical Profile If “Internal Wave” Exists

N_{Seiche} : “Internal Wave’s Period”



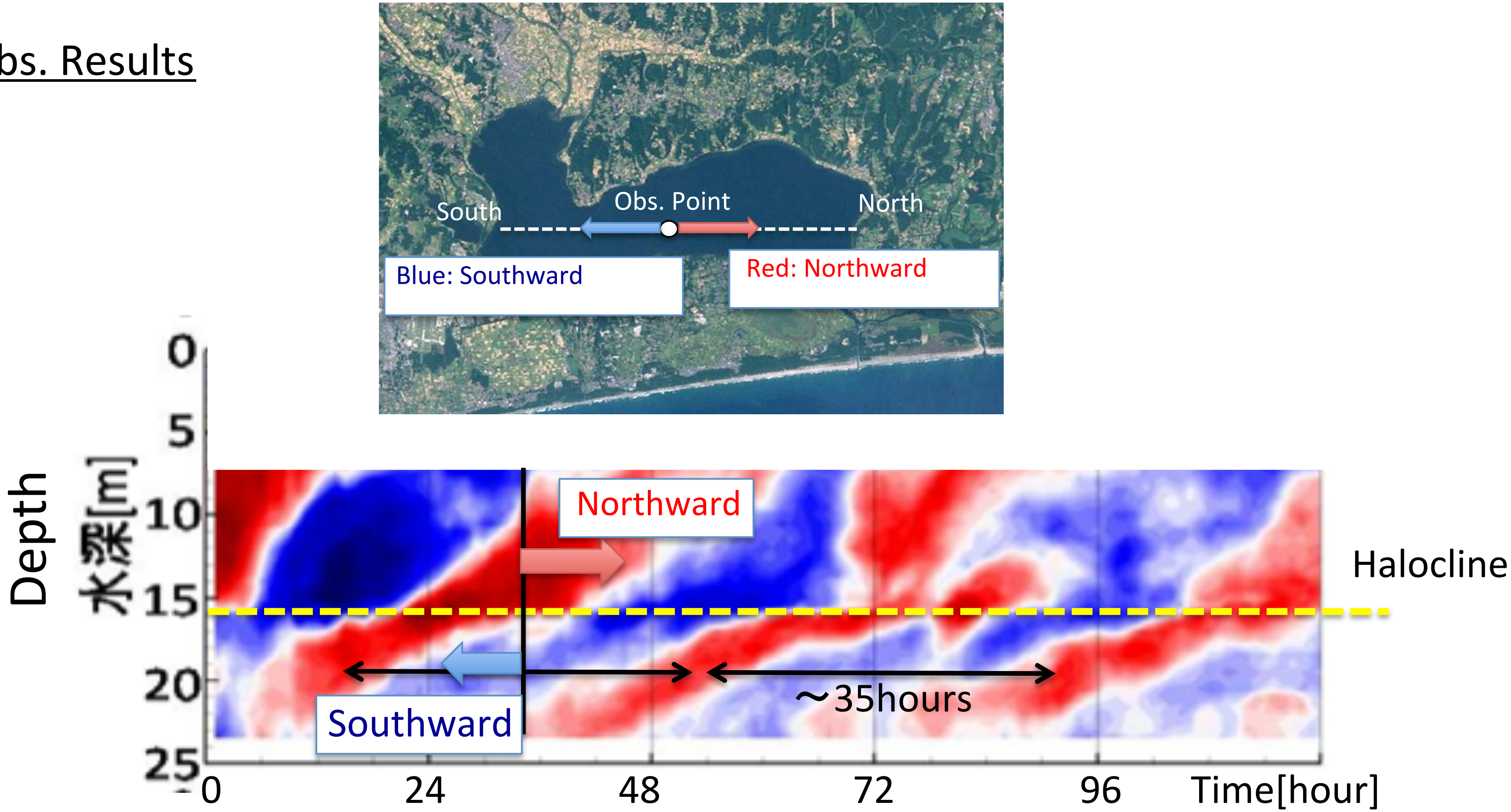
Color Contour of Horizontal Velocity at Central Lake

> After a Half of Period Passed, Flow Directions Change Opposite.

> After a Half of Period Passed Again, Flow Directions Restore Initial Direction.

Namely, Flow Direction is Changing with Period of “Internal Wave”.

Obs. Results



Vertical Profile of Velocity in the North-South Direction.

>Flow Direction Changes Opposite Between Halocline.

>Flow Direction Changes Periodically. (Period : $T_{Obs} \cong 35[hour]$).

➡ Agree With Theoretical Period of “internal Wave” : $T_{Seiche} \cong 32[hour]$

Suggesting There Must be “Internal Wave” Phenomena.

Summary of Dynamics of Stratification Interface Under a Stationary Flow $\left(\mathbf{u}=0\right)$

> Stability of Density Stratification Depends on Vertical Gradient of Density $\dfrac{\partial\rho}{\partial z}$

> When Upper Water Layer is Lighter than Bottom ($\dfrac{\partial\rho}{\partial z}<0$),

- Stratification is Called “Stable (Normal) Stratification”
- Interface Oscillates Stably Keeping Amplitude Constant
- Interface’s Oscillation is Called “Internal Wave / Seiche”.
- Oscillation’s Period is Roughly Evaluated by $N_{Seiche}\cong\dfrac{1}{L}\sqrt{H_B\times g'}$

> When Upper Water Layer is Heavier than Bottom ($\dfrac{\partial\rho}{\partial z}>0$),

- Stratification is Called “Unstable (Inverse) Stratification”
- Displacement of Interface can Grow Endlessly.
- Stratification Can not be Kept Stably.