§2.4 Feature of Dynamics of Density Stratification – Oscillation & Stability of Stratification

① Under a Stationary Flow Field $\left(\mathbf{u}=0\right)$.

— Fundamental Dynamics of Interface in Calm Water Area (Lakes,...).

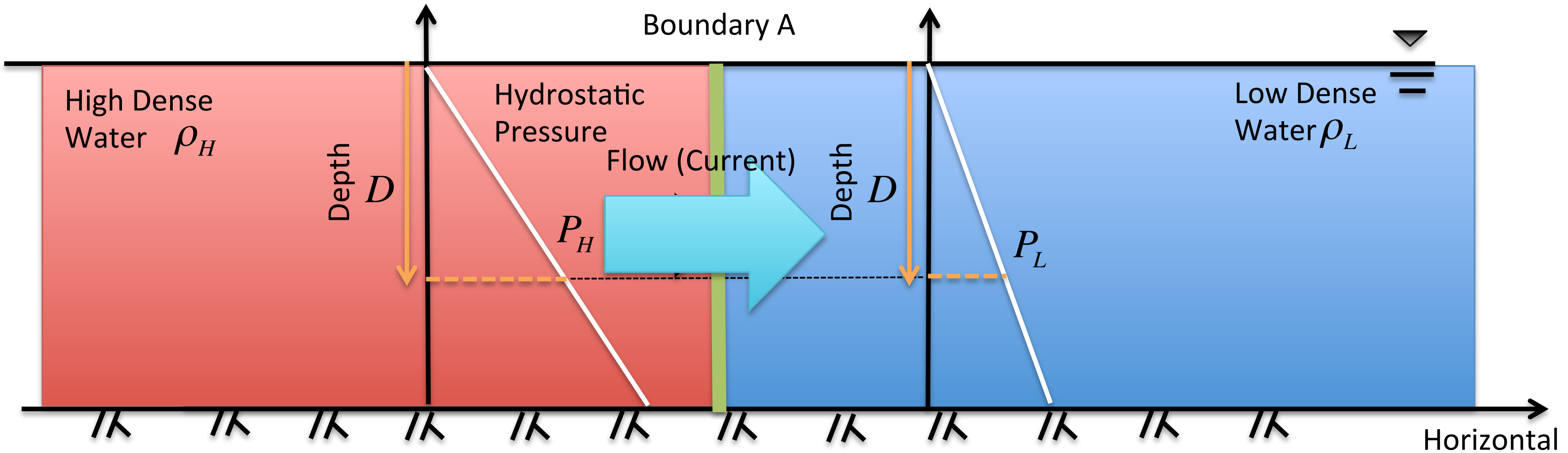
② Under a Non-Stationary Flow Field $\left(\mathbf{u}\neq 0\right)$.

— Water Current Caused by Spatial Density Difference (Density Current).

— Fundamental Dynamics of Interface in Currents Existing Water Area (Rivers,...).

<u>Density Current</u>

> Generally, Horizontal Spatial Change of Density Causes the Current. ➡ <u>Density Current</u>



Hydrostatic Pressure: $P_H = \rho_H D$ **>** $P_L = \rho_L D$

$$\because \rho_H > \rho_L$$

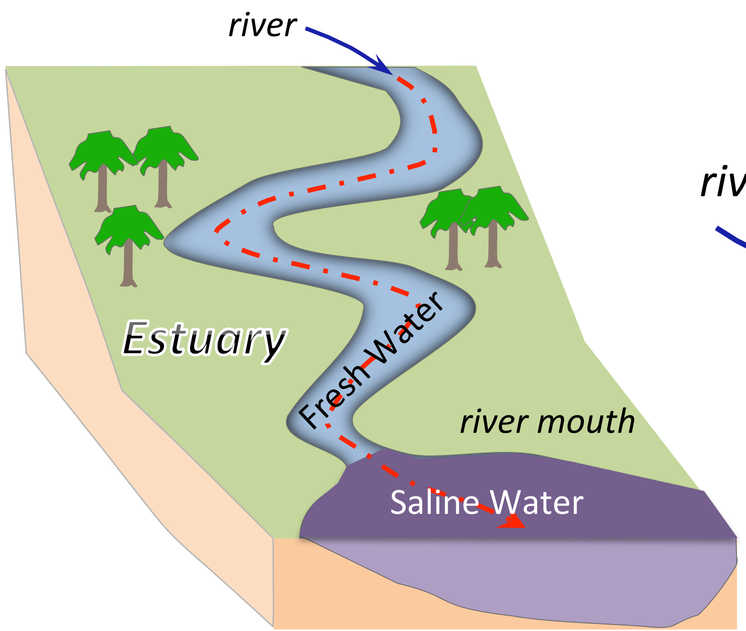
Force on Boundary A; $F_A = P_H - P_L > 0$ (Force Directing +X Affects on A)

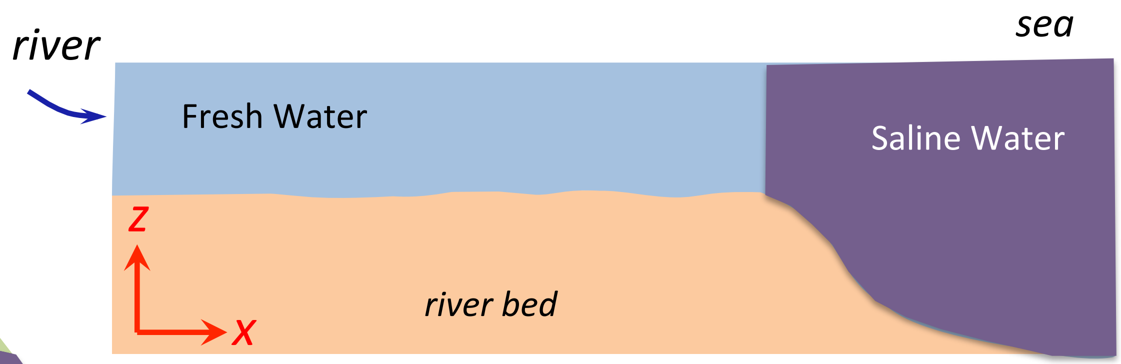
<u>Water Around Boundary is Accelerated & Horizontal Flow (Current) is Generated.</u>

Density Current can Occur in Various Water Fields,

Most Significant Case is "Salt Wedge Phenomena" in Estuary.

Estuary : Water Regions Where Fresh and Saline (Sea) Water Coexist.
(Typical Regions : Around the River Mouth)

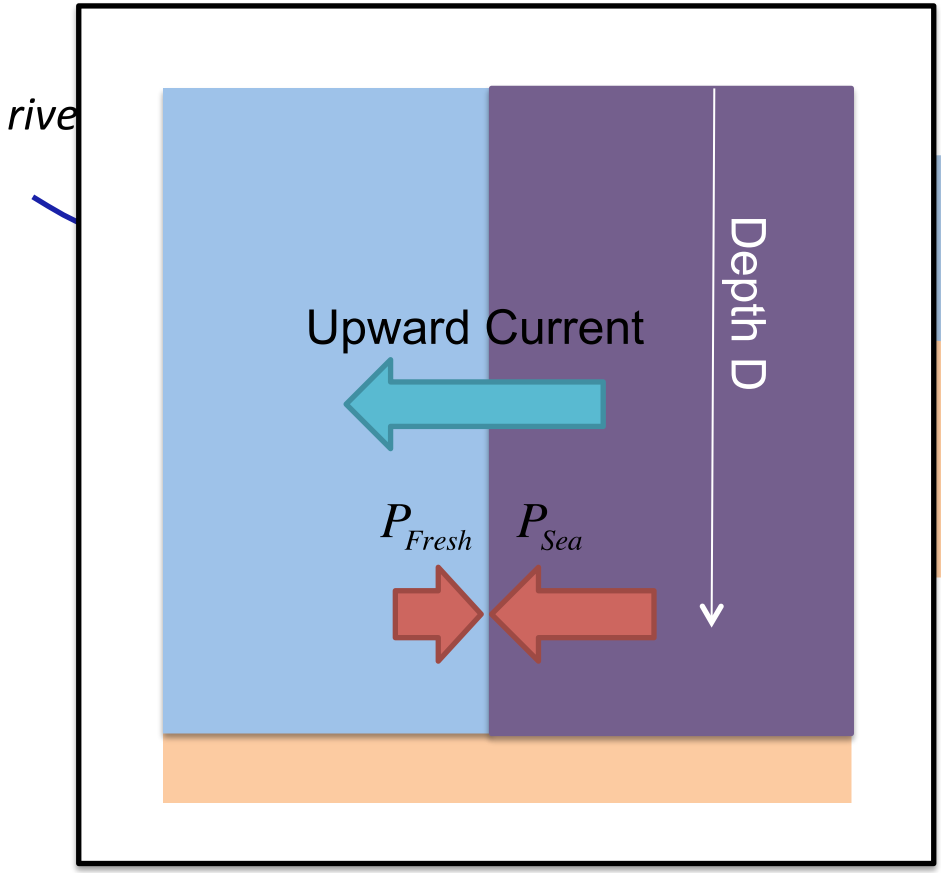


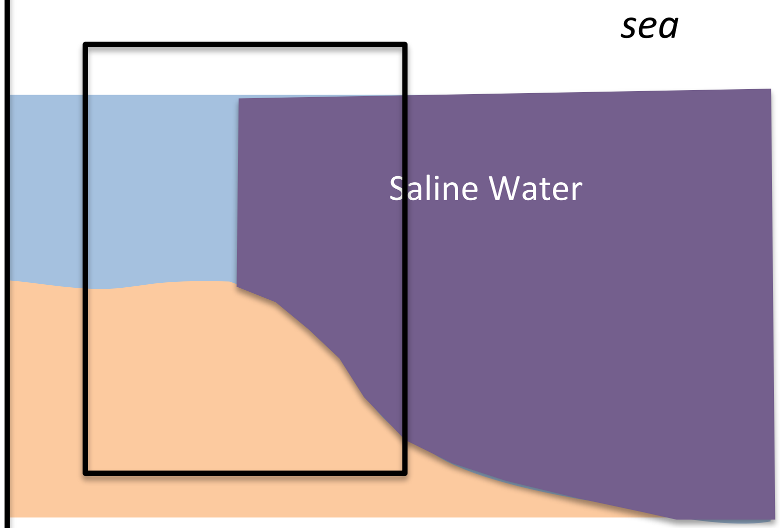


"Salt Wedge" in Estuary

Density: Sea Water (Salinity ∼ 35[‰]) $\rho_{Sea} \cong 1{,}028[kg/m^3]$
Fresh Water (Salinity ∼0[‰]) $\rho_{Fresh} \cong 1{,}000[kg/m^3]$

➡ Sea Water is Heavier : $\Delta\rho \cong \rho_{Sea} - \rho_{Fresh} \cong 28[kg/m^3]$





Hydrostatic Pressure :

$P_{Sea} = \rho_{Sea} g D$ (Sea side)

$P_{Fresh} = \rho_{Sea} g D$ (River side)

Because $P_{Sea} > P_{Fresh}$, Upward Current is Generated.

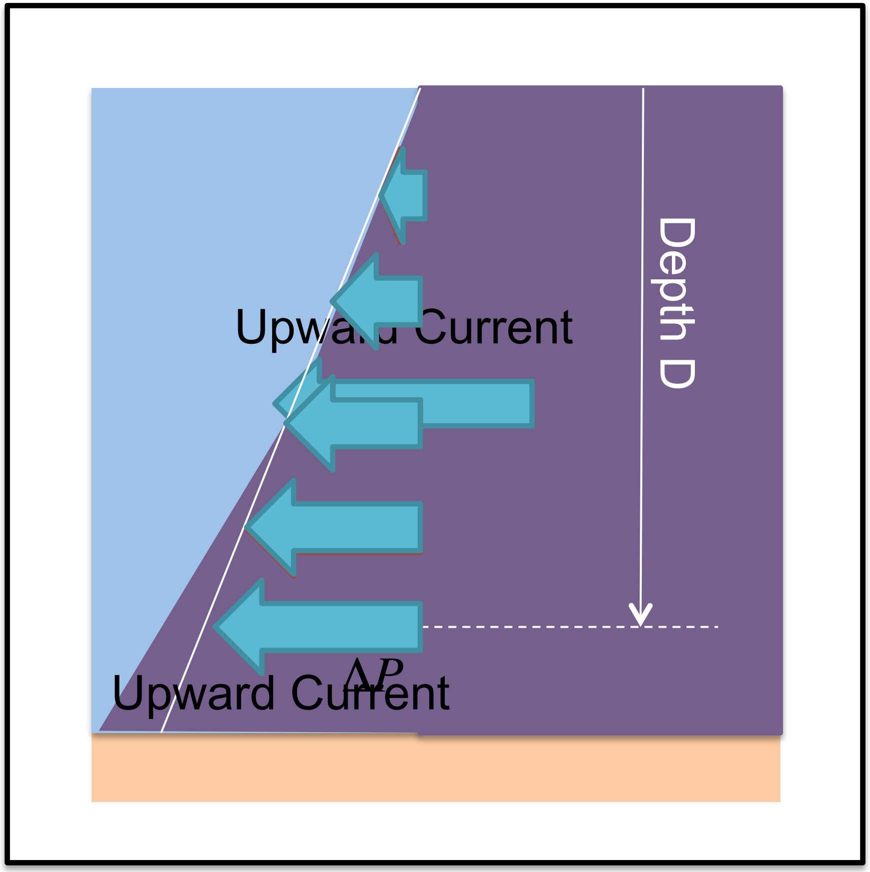
“Salt Wedge” in Estuary

> Upward Current is Generated due to Difference of Hydrostatic Pressure.

> Difference of Hydrostatic Pressure:

$$\Delta P = P_{Sea} - P_{Fresh} = \rho_{Sea} g D - \rho_{Fresh} g D = g\left(\rho_{Sea} - \rho_{Fresh}\right) \times D$$

Proportion to “Depth” D.

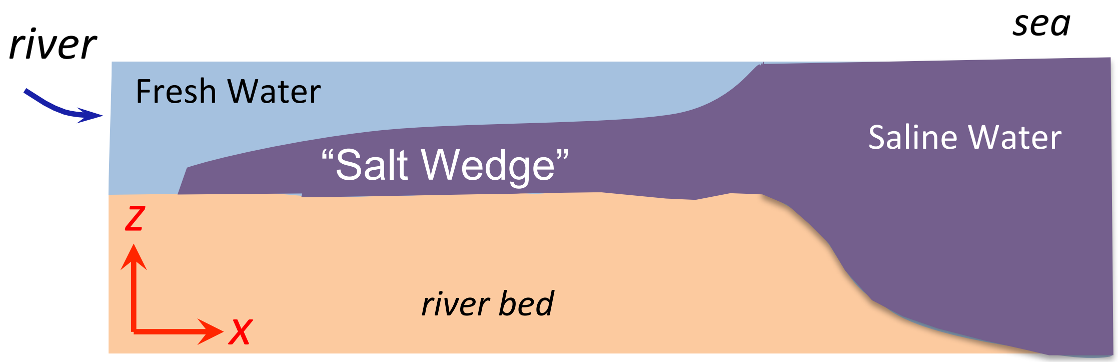


> Bottom Water is Accelerated More Strongly than Upper Water.

> Bottom Sea Water Will be the First to Intrude to Upstream of River Channel.

Around the River-Bed, Sea Water Intrudes to Upstream.

In the River Channel of Estuary, Two Layers, Which Consists of Bottom Saline Water and Upper Fresh Water, is Usually Generated.



After a Shape of Bottom Saline Water Layer, This Phenomena is Called "Salt Wedge".

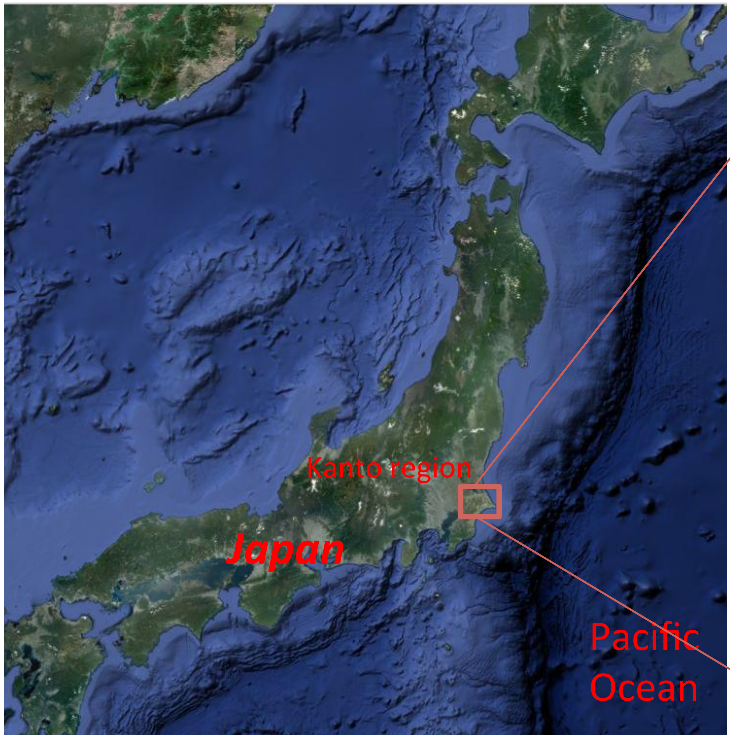
Obs. of Salt Wedge in Tone River, Japan

>Tone River : 2nd longest River in Japanese Main Island.

>Barrage was Constructed at 18 km upstream from river mouth.

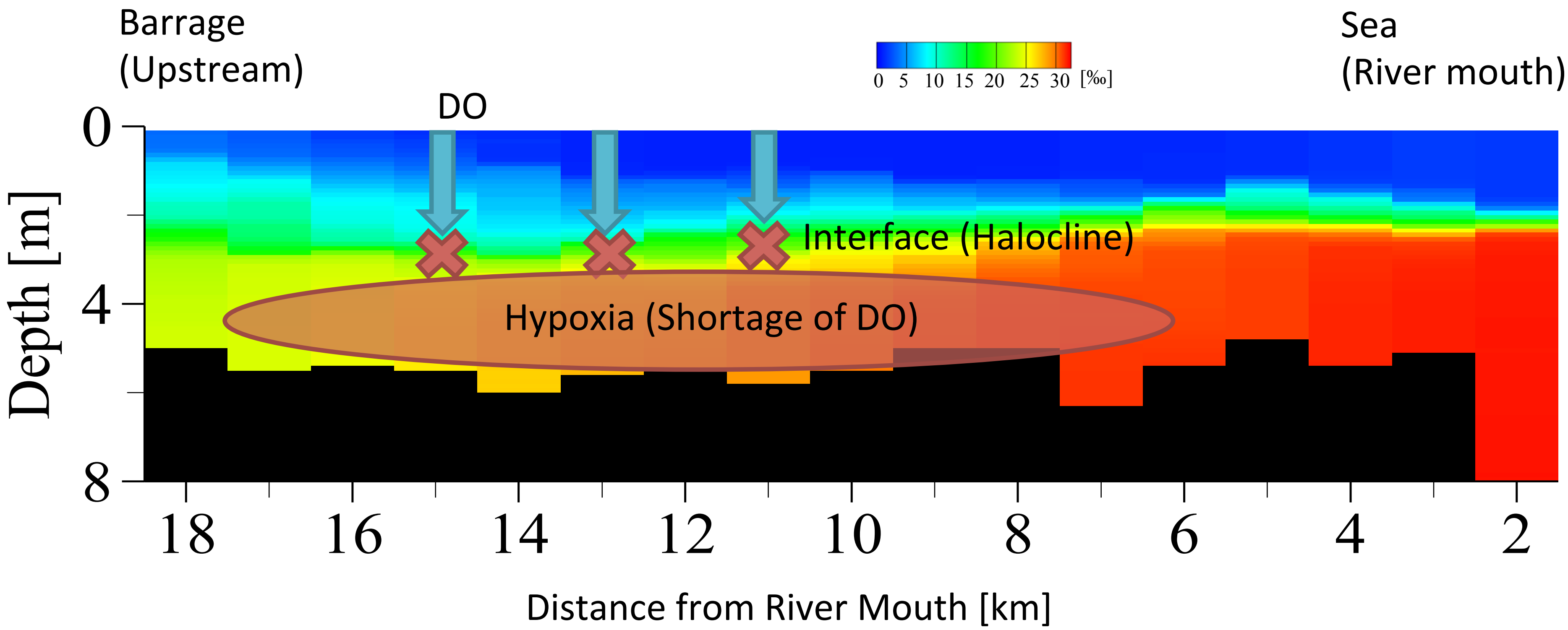
>Except It is Rainy, Gates are almost Closed & Small Amount of Fresh Water is Discharged.

>Salt Wedge Easily Stays in the River Channel.



Obs. of Salt Wedge in Tone River, Japan

>Observed Salinity Distribution Along the River Channel.



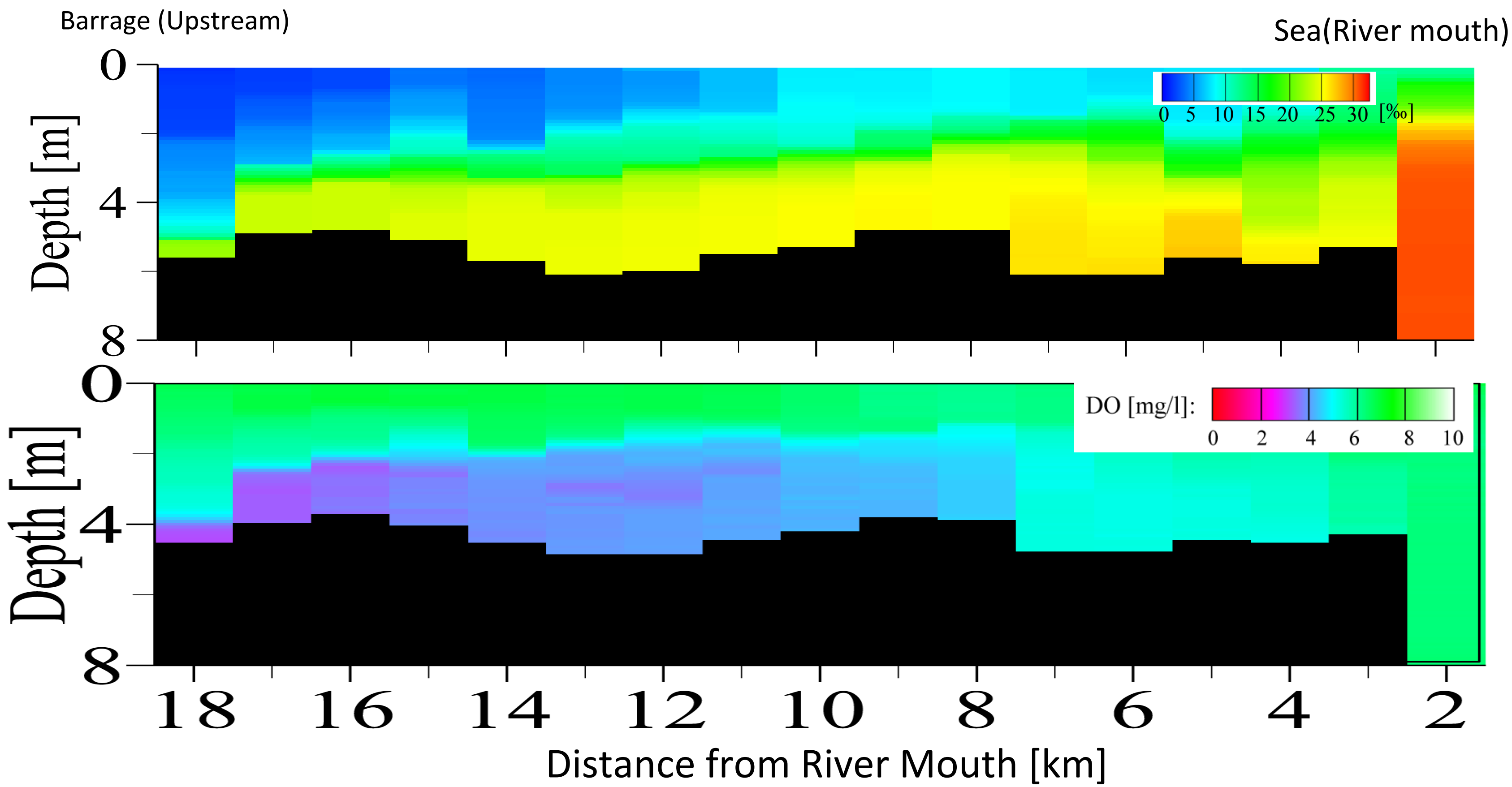
> Density Difference is Relatively Larger than Thermocline Occurred in Lakes.

> Vertical Transportation of DO is Strongly Suppressed by Stratification (Interface of Salt Wedge)

> The Bottom Saline Water Layer is in Danger of Shortage of DO (Hypoxia).

Obs. of Salt Wedge in Tone River, Japan

>Observed Salinity & DO Distribution Along the River Channel.



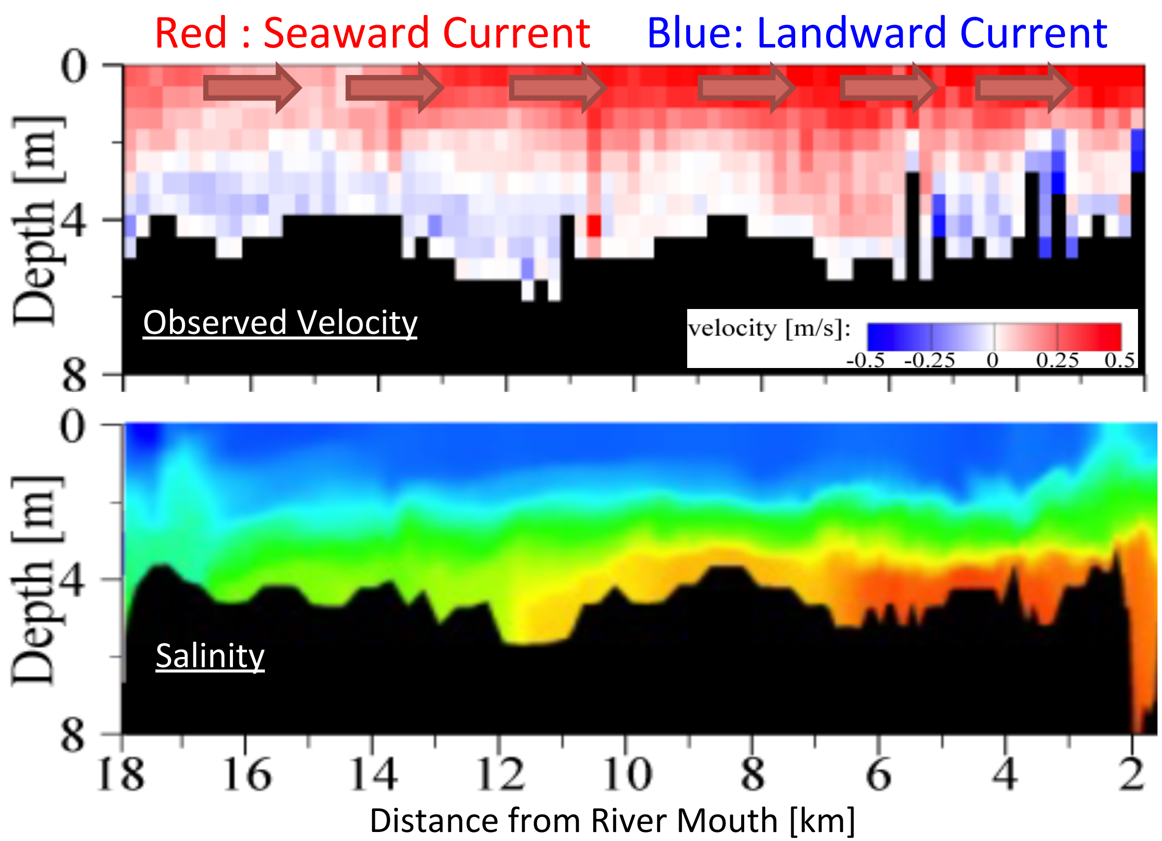
> In the Bottom Saline Water Layer, DO Reduces to 3〜4 [mg/*l*] & Hypoxia Occurs.

(cg. According to Guideline of Water Quality of Japanese Ministry,
Under DO<4[mg/l], There is the Possibility of Deaths of Fishes)

> Salt Wedge is Important to Preserve Health of Estuary's Water Quality.

In Actual Estuary, “Salt Wedge” Always Moves

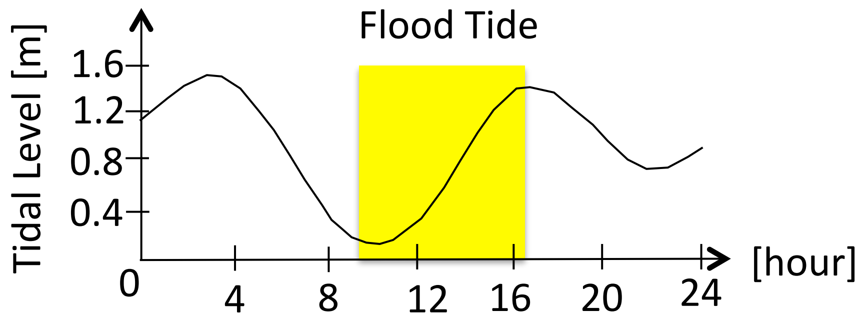
When Large Amount of Fresh Water is Discharged from Upstream Barrage,



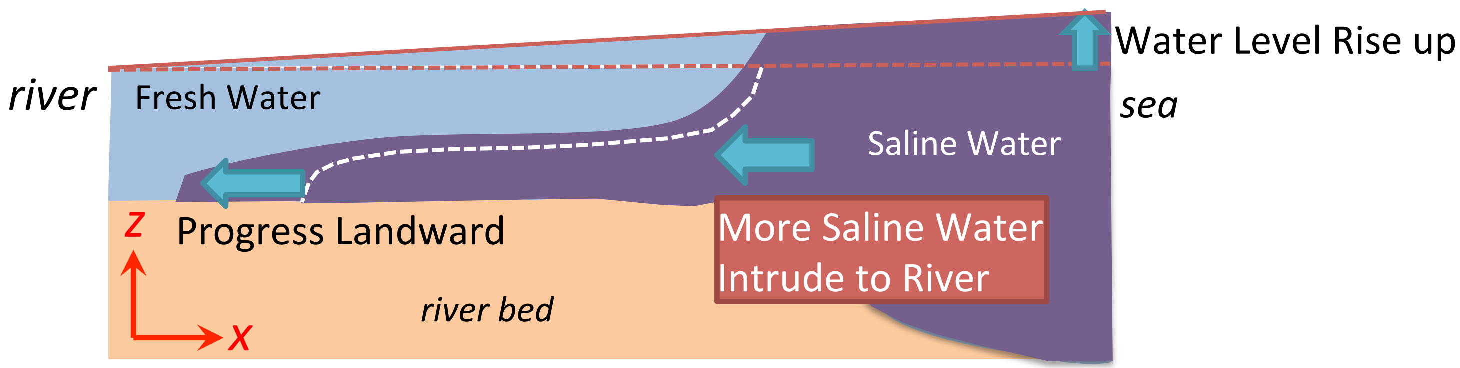
- ➢ Because the Discharged Fresh Water is Lighter than Saline Water, Fresh Water Flows Upper Layer.
- ➢ Seaward Current Appears Only in Upper Layer.

In Actual Estuary, "Salt Wedge" Always Moves

- Water Level at Sea Changes Periodically due to the Tide.



- When Flood Tide in that Water Level Rises up,

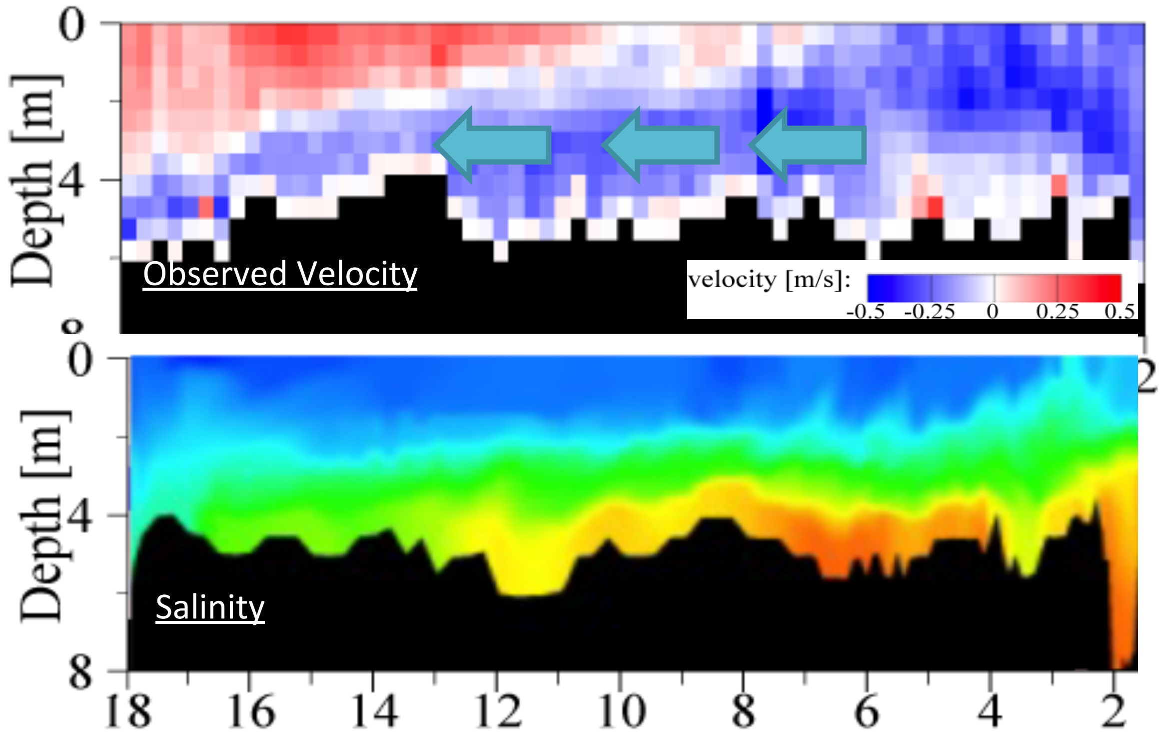


- Water Level of Sea is Higher than that of River.
- Because More Saline Water Intrude to River, "Salt Wedge" is Pushed & is Progressing Landward.

In Actual Estuary, “Salt Wedge” Always Moves

Observed Velocity at “Flood Tide”

Red : Seaward Current   Blue: Landward Current

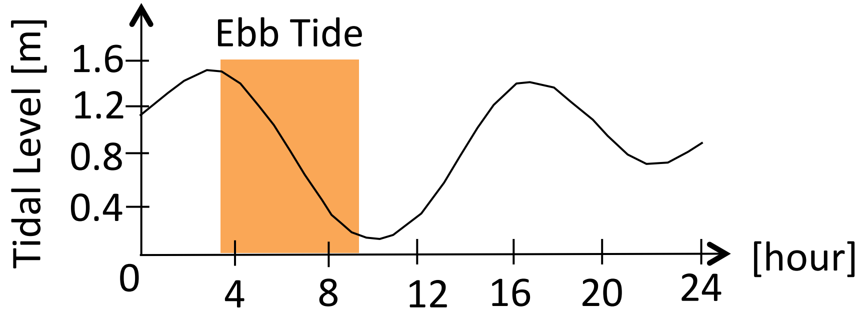


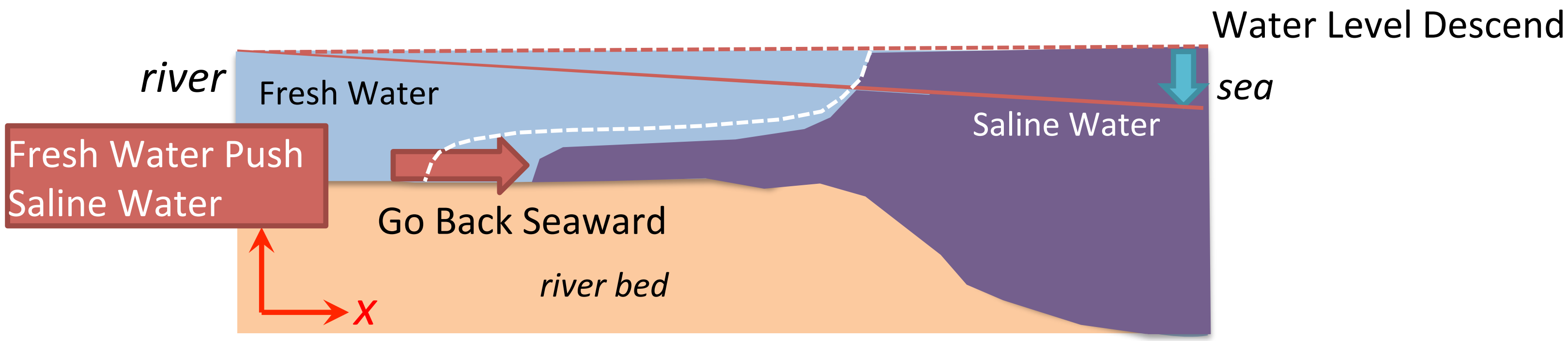
Distance from River Mouth [km]

➢ Landward Current Appears Selectively in the Bottom Water Layer (“Salt Wedge”).

In Actual Estuary, "Salt Wedge" Always Moves

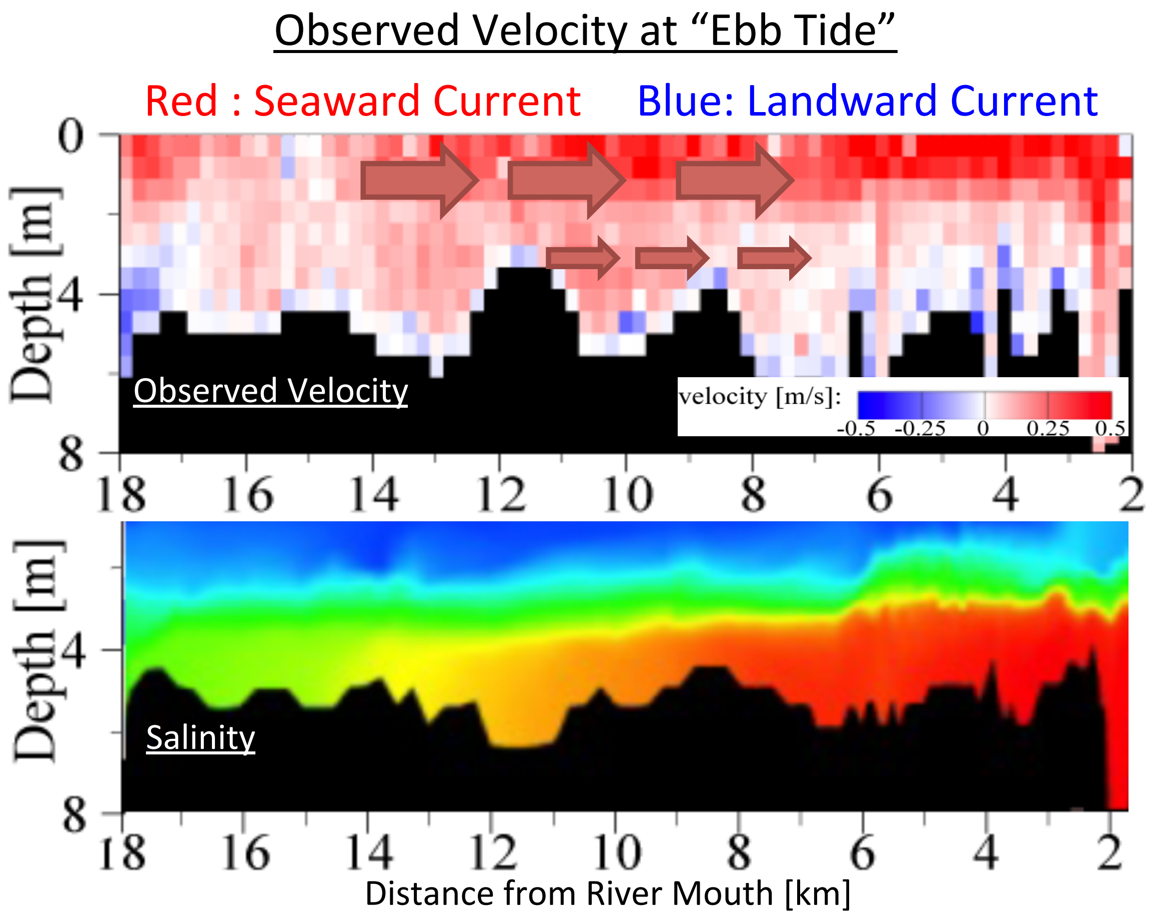
- When Ebb Tide in that Water Level Descend down,





- Water Level at River Channel is Higher than that of Sea.
- Because Fresh Water Tends to Flow Seaward & Push the Saline Water, "Salt Wedge" is Going Back Seaward.

In Actual Estuary, "Salt Wedge" Always Moves



- Seaward Current Appears Both in Upper & Bottom Layer.
- Due to the Bottom Friction, Current in the Bottom Layer is made weaker.

As Generally Speaking,

>Flow with “Salt Wedge” is Complex.

>Generally, Flow is not Stationary ($u \neq 0$).

On the Other hand,

>In the Estuary, there is the possibility that Bottom Water Becomes Hypoxia.

<u>In the Case of Stationary Flow Filed ($u = 0$)</u>

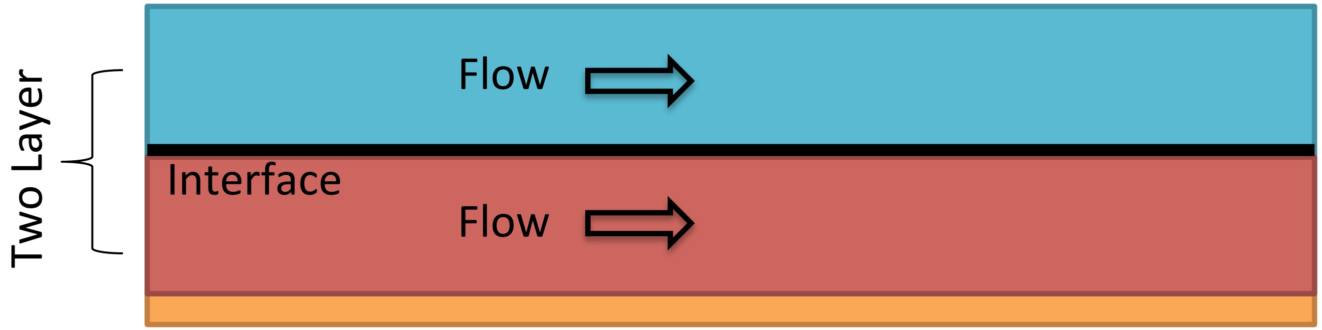
>”Whether Stratification can be Kept Stably or not” Mainly Depends only on the Vertical Spatial Gradient of Density $\partial \rho / \partial z$.

<u>In the Case of Non-Stationary Flow Filed ($u \neq 0$)</u>

>”Whether Stratification can be Kept Stably or not” also Depends on the Magnitude of Flow.

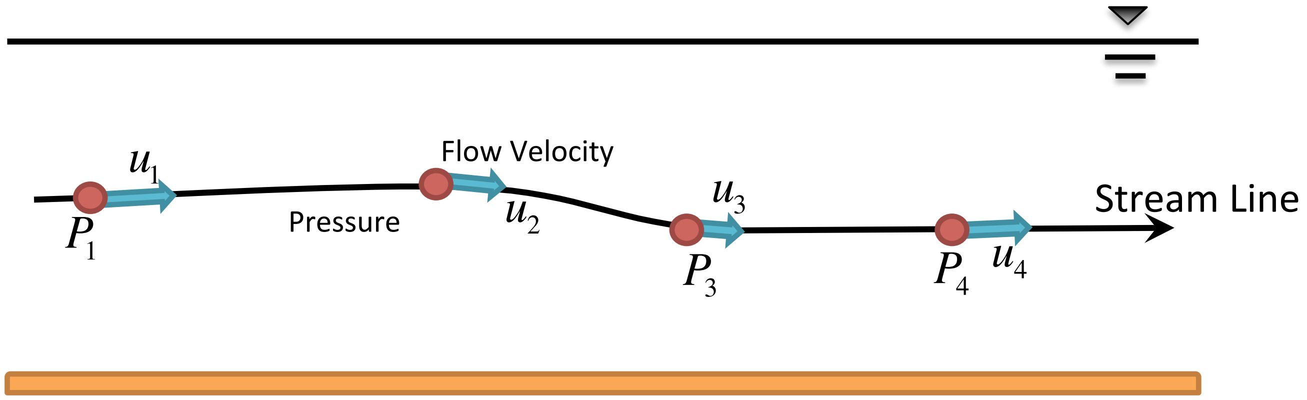
Hereafter,

>Investigate General Behavior of Interface of Two Layers Under non-Stationary Flow Field.



Through the Investigation, “Bernoulli‘s principle” will be Active

For a Incompressible Fluid (Water), When Extra forces and Viscosity is Relatively Weak,



Although Flow Velocity u & Pressure P can Change along Stream Line,

u & P Must Satisfy the Following Relation;

$$\frac{1}{2}u_1^2 + \frac{P_1}{\rho} = \frac{1}{2}u_2^2 + \frac{P_2}{\rho} = \frac{1}{2}u_3^2 + \frac{P_3}{\rho} = \frac{1}{2}u_4^2 + \frac{P_4}{\rho} = const.$$

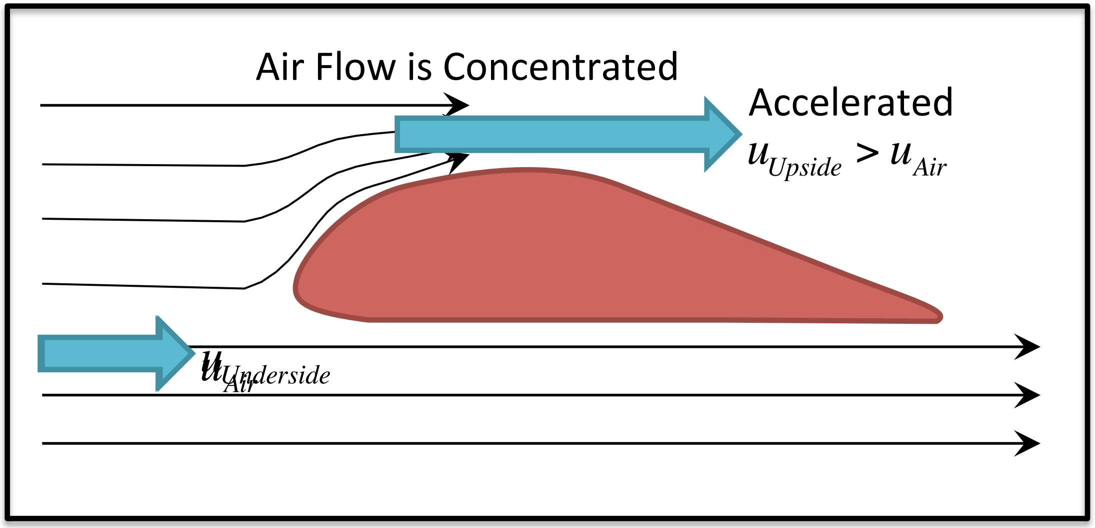
ρ :Density

Bernoulli’s Principle

Even if You Know the “Bernoulli’s Principle”, You are Utilizing in Daily Life.

Airplane

Cross Section of Wing



>Generally, Wing Has a Convex Shape.

>At the Underside of Wing, Air can Flows without Resistance
& Flow Velocity is not Changed; $u_{Underside} = u_{Air}$

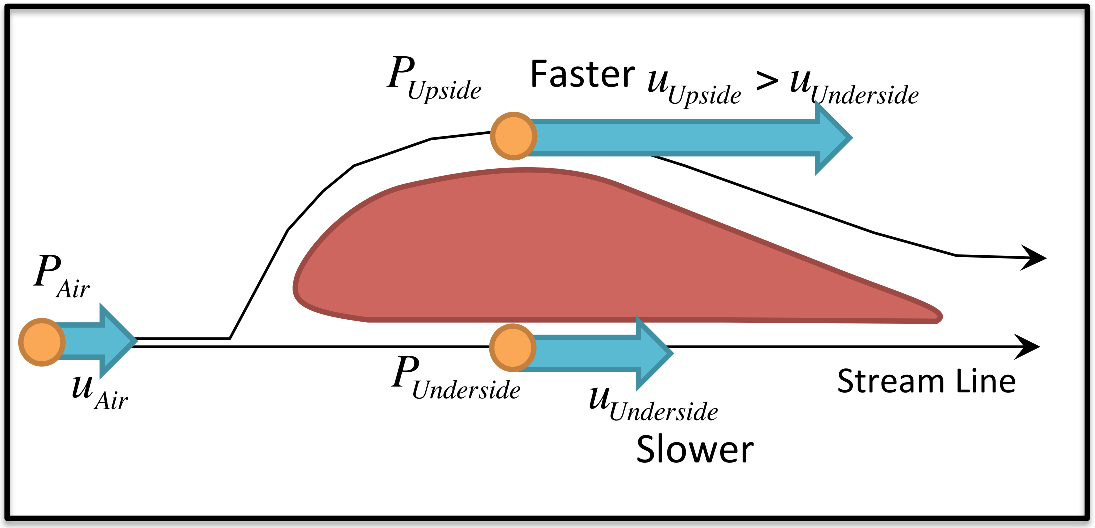
>At the Upside of Wing, Air Flow is Concentrated by Wing.

>Air Flow is Pushing Each Other
& Flow Velocity at Upside is Strongly Accelerated; $u_{Upside} > u_{Underside} = u_{Air}$

Even if You Know the "Bernoulli's Principle", You are Utilizing in Daily Life.

Airplane

Cross Section of Wing



> Assuming at Upstream of Stream line, Pressure and Velocity are Given by P_{Air} & u_{Air} .

> According to "Bernoulli's Principle",

For the Underside: $$\frac{1}{2}u_{Air}^2 + \frac{P_{Air}}{\rho_{Air}} = \frac{1}{2}u_{Underside}^2 + \frac{P_{Underside}}{\rho_A}$$

Pressure on the Underside : $$P_{Underside} = \frac{\rho_A}{2}\left(u_{Air}^2 - u_{Underside}^2\right) + P_{Air}$$

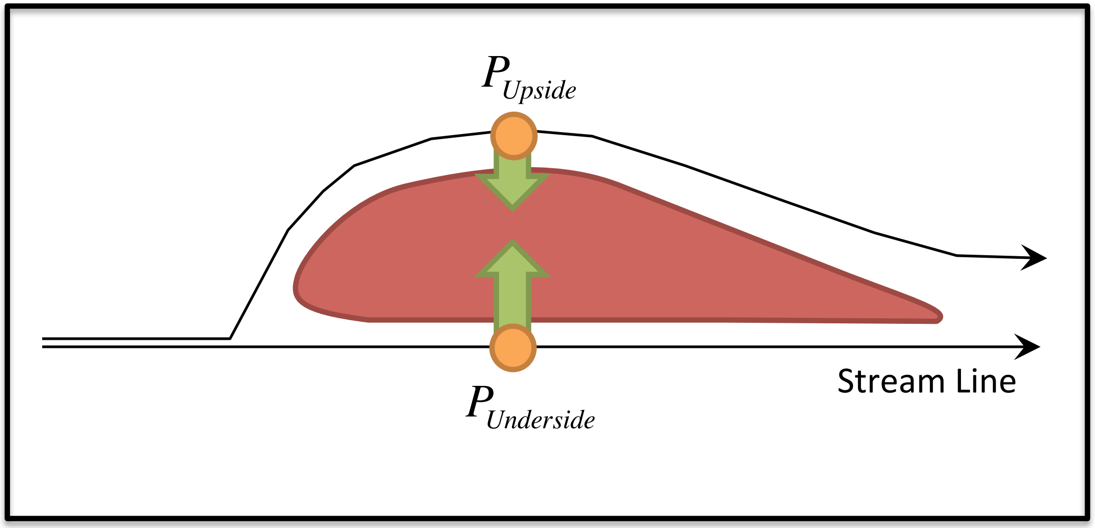
For the Upside: $$\frac{1}{2}u_{Air}^2 + \frac{P_{Air}}{\rho_{Air}} = \frac{1}{2}u_{Upside}^2 + \frac{P_{Upside}}{\rho_A}$$

Pressure on the Underside : $$P_{Upside} = \frac{\rho_A}{2}\left(u_{Air}^2 - u_{Upside}^2\right) + P_{Air}$$

Even if You Know the "Bernoulli's Principle", You are Utilizing in Daily Life.

Airplane

Cross Section of Wing



> On Both side of Wing, Pressure Affects in Vertical Direction.

> $P_{Underside}$ Affects to Push the Wing Upward & P_{Upside} Affects to Push Downward.

> Net Force which the Wing is Affected; $F_{Wing} = P_{Underside} - P_{Upside}$

> Substituting Each Pressure; $F_{Wing} = \frac{\rho_A}{2}\left(u_{Upside}^2 - u_{Underside}^2\right)$

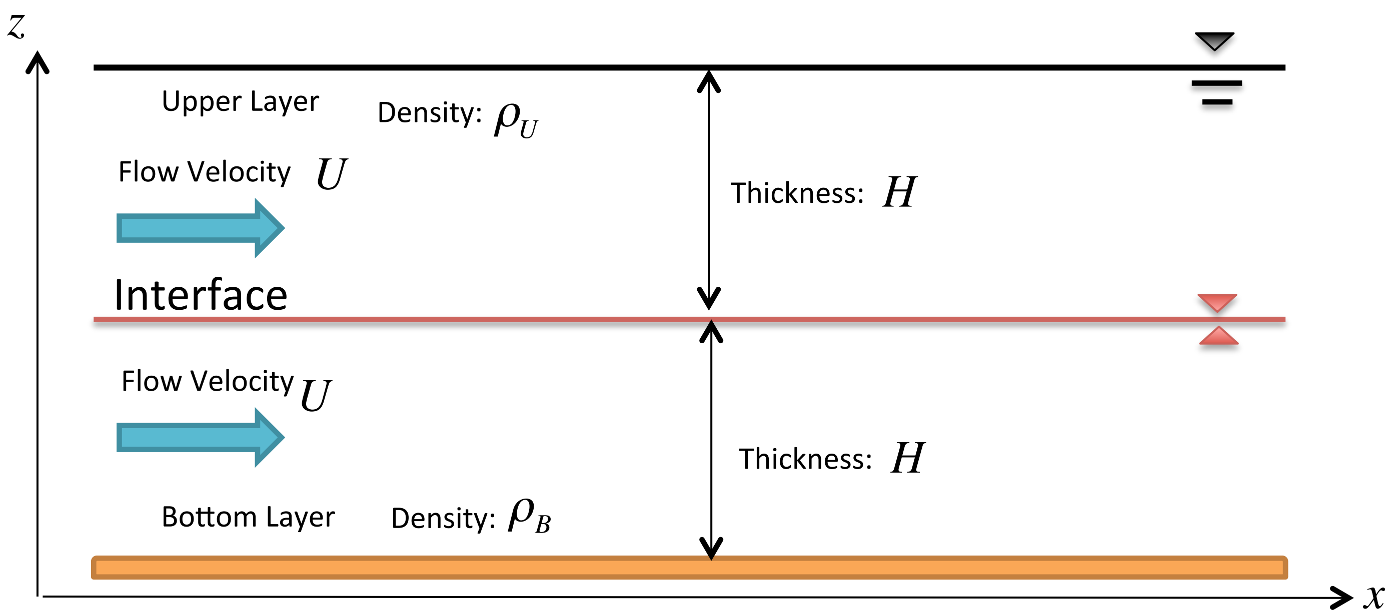
> Flow Speed at Upside is Faster ; $u_{Upside} > u_{Underside}$

> Force on the Wing ; $F_{Wing} = \frac{\rho_A}{2}\left(u_{Upside}^2 - u_{Underside}^2\right) > 0$ ⇒ Provide the Lifting Force & Air Plane can Fly.

Investigation of Behavior of Interface with non-Stationary Flow Field using Bernoulli's Principle"

>For Simplicity, We Assume,

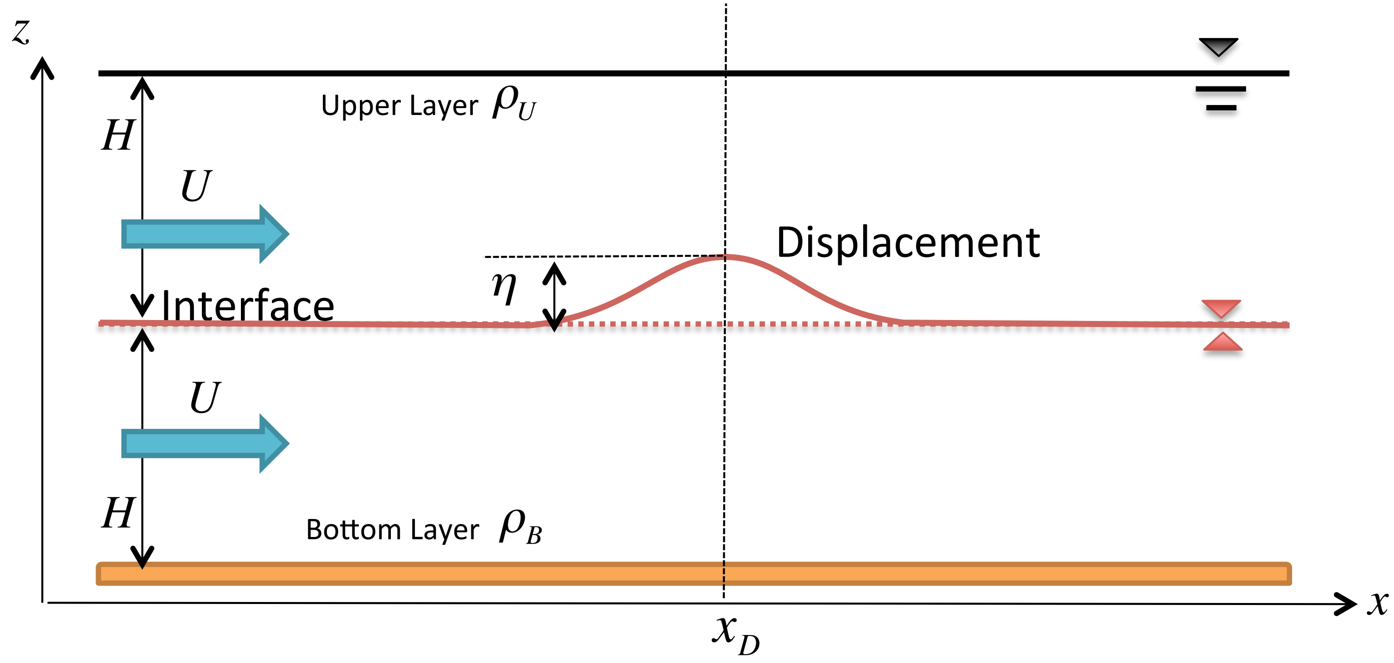
- Flow consists of Two Layers Which have Different Density : $\rho_U \neq \rho_B$
- Thickness of Each Layer is Equal and Given by H .
- Velocity of Inflow at Upstream is Equal and Given by U .
- Water Surface is Kept to be Flat (Not-Changed)



Investigation of Behavior of Interface with non-Stationary Flow Field

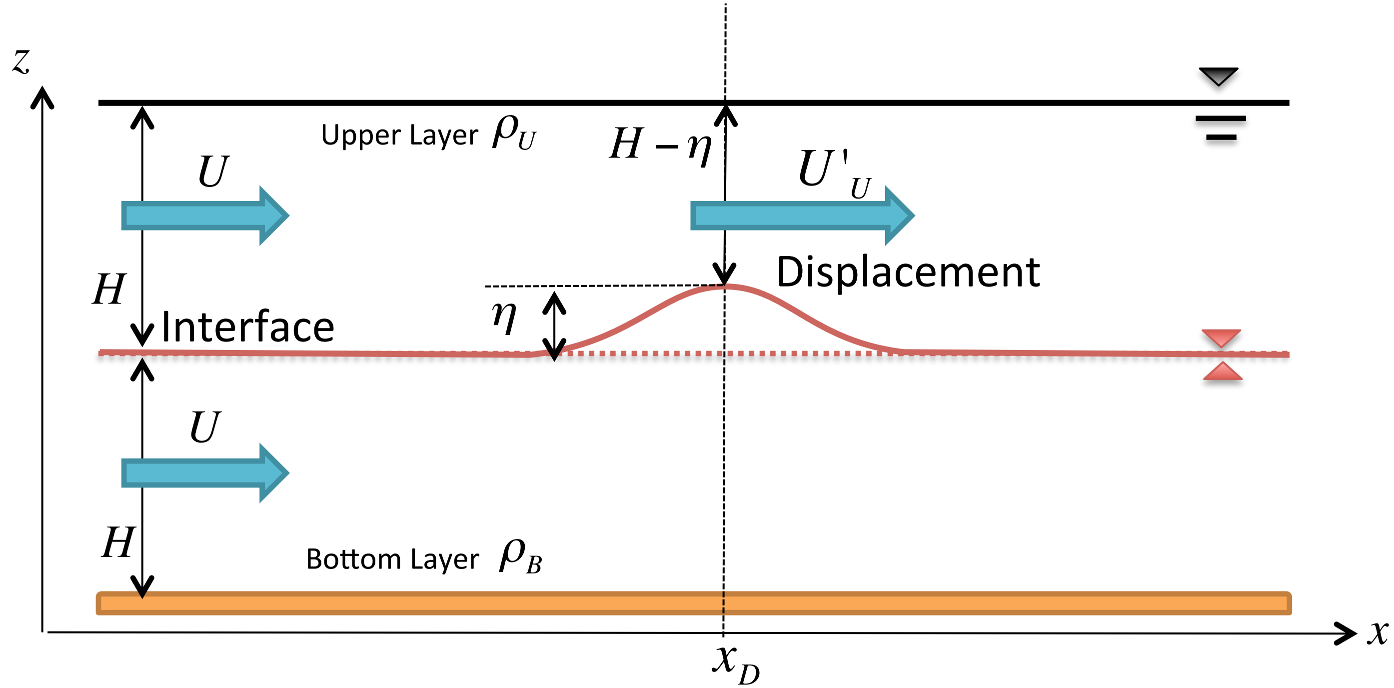
using Bernoulli's Principle"

>Under these Assumptions, Suppose that Small Displacement Occurs on the Interface.



Investigation of Behavior of Interface with non-Stationary Flow Field

using Bernoulli's Principle"



>Thickness of Upper Layer where Displacement Occurs (x_D) Changes to $H-\eta$.

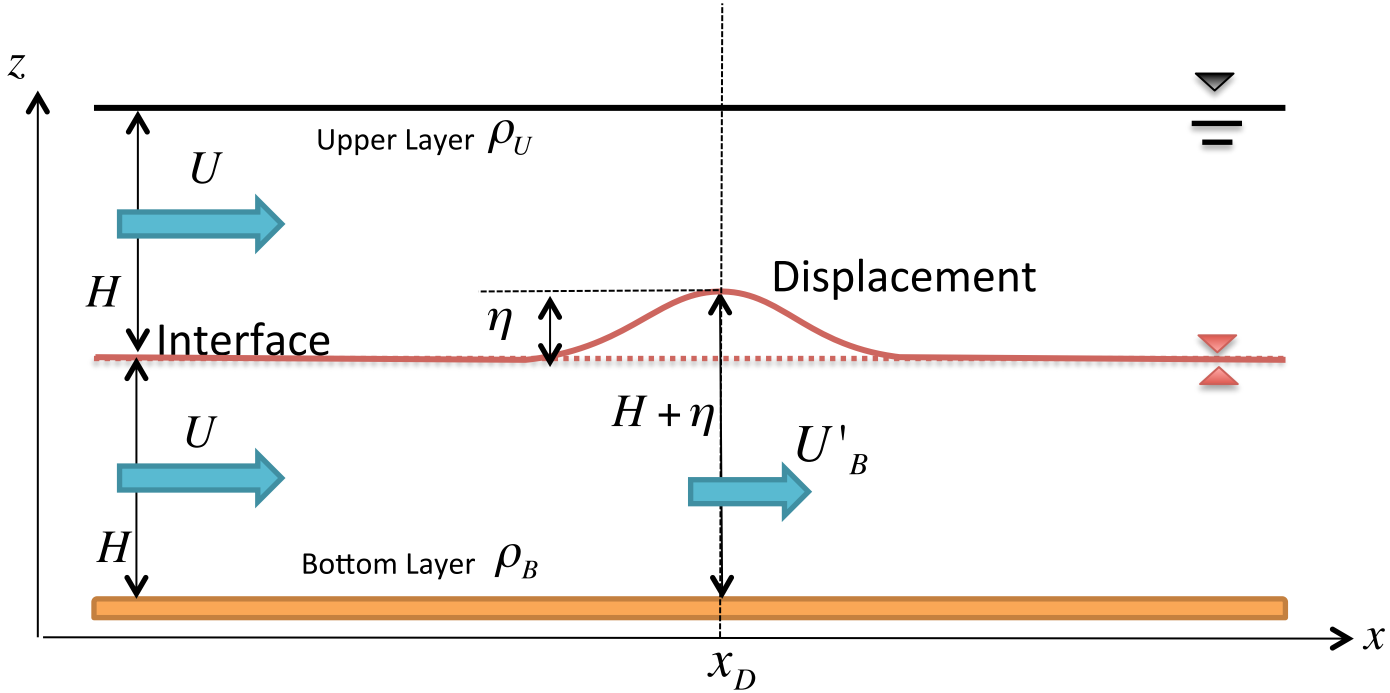
>Suppose the Flow Velocity at x_D ; U'_U

Because the Flux in the Upper Layer Must be Conserved,

$$U \times H = U'_U \times (H-\eta) \quad \Rightarrow \quad \therefore U'_U = \frac{H}{H-\eta} U$$

Investigation of Behavior of Interface with non-Stationary Flow Field

using Bernoulli's Principle"



>Thickness of Bottom Layer where Displacement Occurs (x_D) Changes to $H+\eta$.

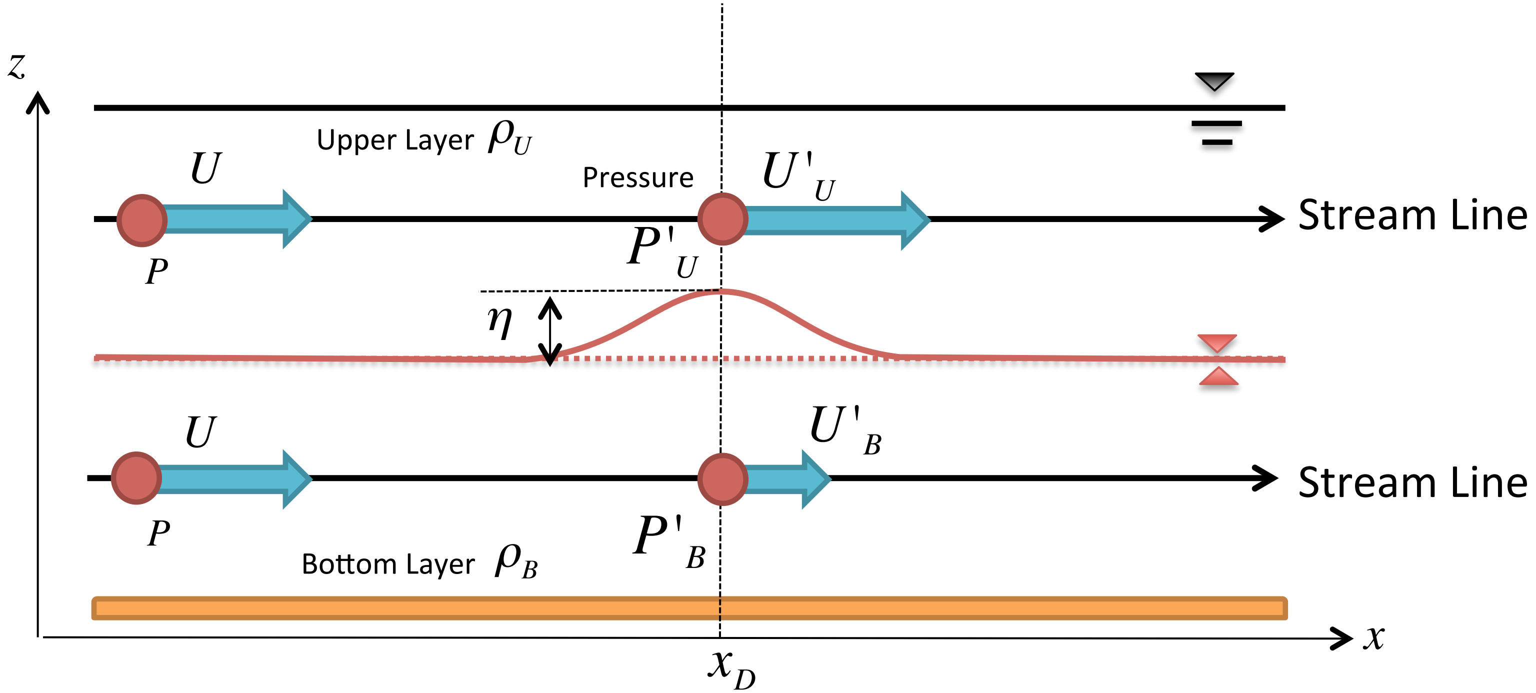
>Suppose the Flow Velocity at x_D ; U'_B

Because the Flux in the Bottom Layer Must be Conserved,

$$U \times H = U'_B \times (H+\eta) \quad \Rightarrow \quad \therefore U'_B = \frac{H}{H+\eta} U$$

Investigation of Behavior of Interface with non-Stationary Flow Field

using "Bernoulli's Principle"

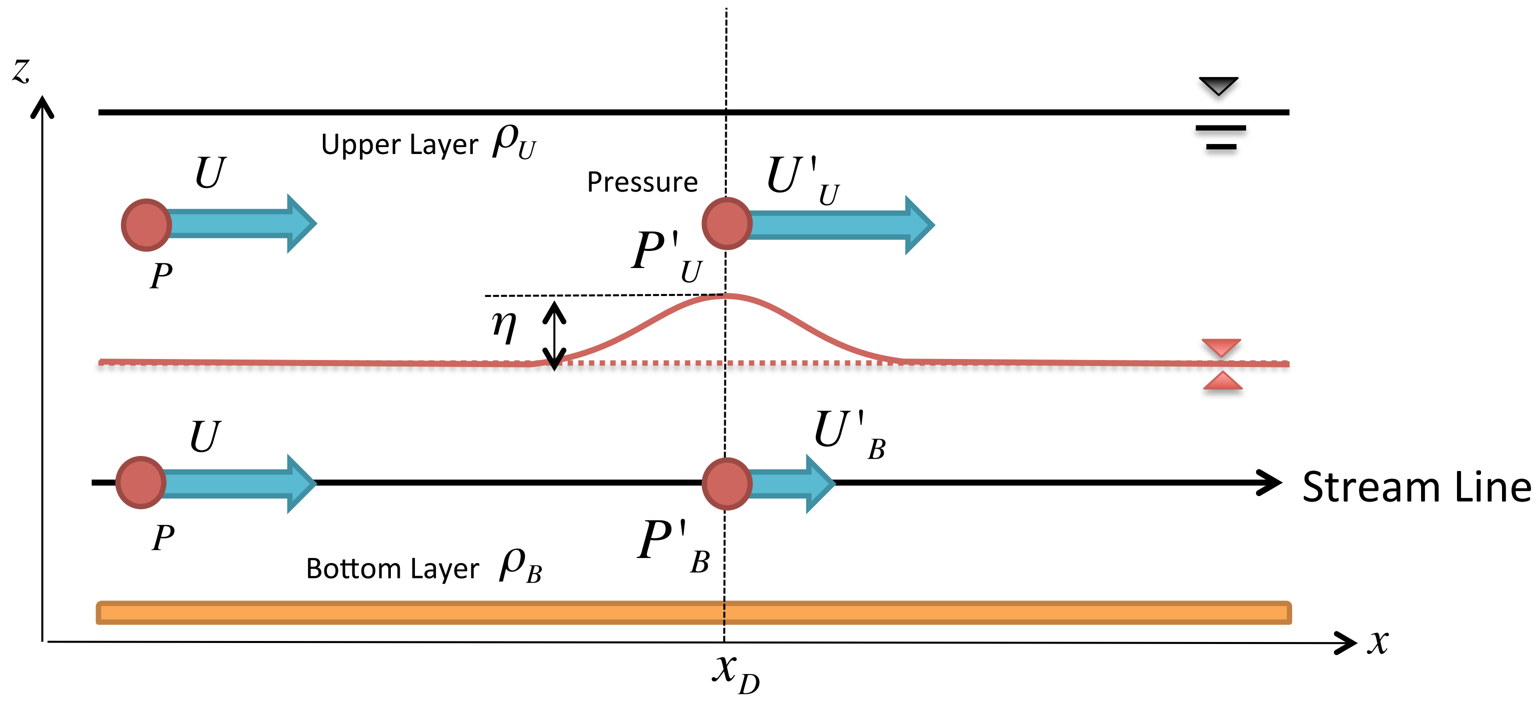


>Assuming the Pressure at Upstream is Same ; P

>By Considering Stream Line in Upper Layer and Applying the "Bernoulli's Principle"

Upper Layer: $$\frac{1}{2}U^2 + \frac{P}{\rho_U} = \frac{1}{2}(U'_U)^2 + \frac{P'_U}{\rho_U} \Rightarrow P'_U = \frac{\rho_U}{2}\left(U^2 - (U'_U)^2\right) + P$$

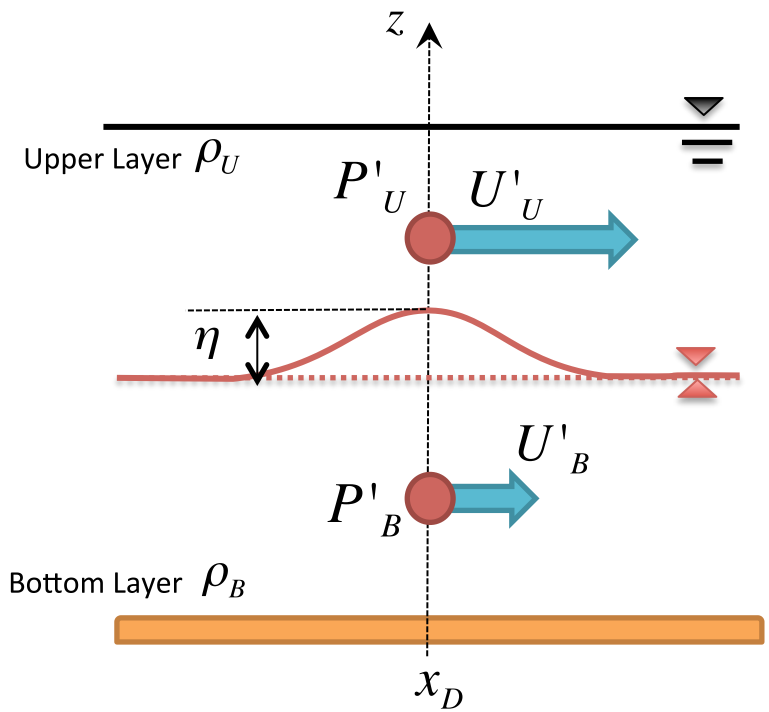
Investigation of Behavior of Interface with non-Stationary Flow Field using "Bernoulli's Principle"



>By Considering Stream Line in Bottom Layer and Applying the "Bernoulli's Principle"

Bottom Layer: $\frac{1}{2}U^2 + \frac{P}{\rho_B} = \frac{1}{2}(U'_B)^2 + \frac{P'_B}{\rho_B} \Rightarrow P'_B = \frac{\rho_B}{2}\left(U^2 - (U'_B)^2\right) + P$

Investigation of Behavior of Interface with non-Stationary Flow Field using "Bernoulli's Principle"



Summary of Pressure

$$P'_U = \frac{\rho_U}{2}\left(U^2 - (U'_U)^2\right) + P$$

$$P'_B = \frac{\rho_B}{2}\left(U^2 - (U'_B)^2\right) + P$$

Substituting Explicit Form of U for Above Eqs.

$$U'_U = \frac{H}{H-\eta}U$$

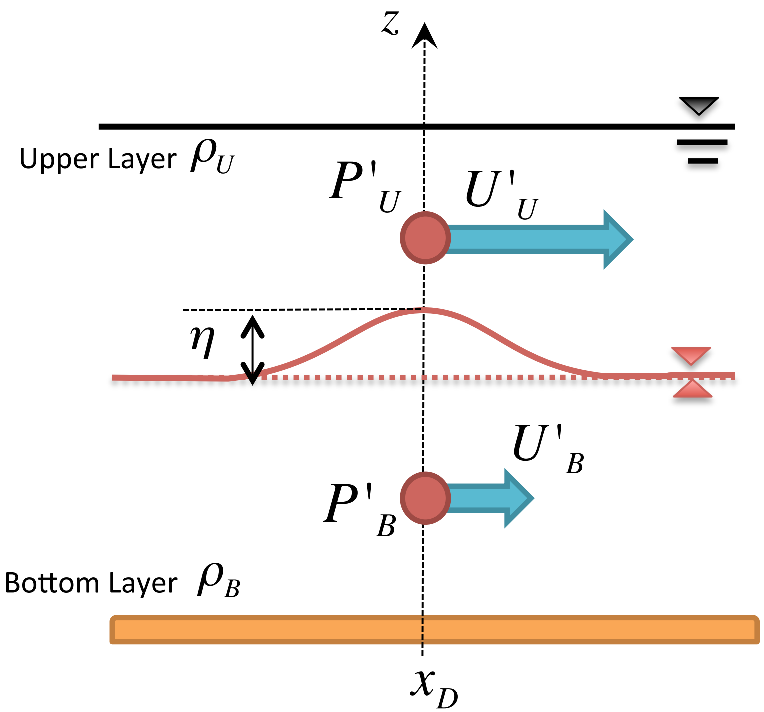
$$U'_B = \frac{H}{H+\eta}U$$

Pressure of Upper Layer: $$P'_U = \frac{\rho_U U^2}{2}\left(1 - \frac{H^2}{(H-\eta)^2}\right) + P$$

Pressure of Bottom Layer: $$P'_B = \frac{\rho_B U^2}{2}\left(1 - \frac{H^2}{(H+\eta)^2}\right) + P$$

Investigation of Behavior of Interface with non-Stationary Flow Field

using "Bernoulli's Principle"



Assuming Displacement is Small: $\eta << H$,

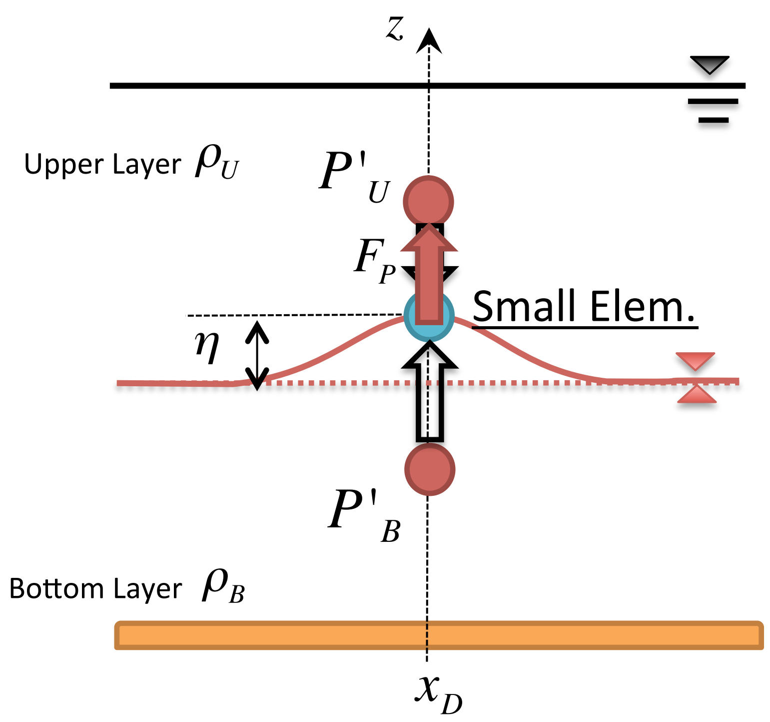
$$\frac{H^2}{(H \pm \eta)^2} \cong \left(1 \mp 2\frac{\eta}{H}\right)$$

Pressures can be Approximated to Simple Form;

$$P'_U = -\rho_U U^2 \frac{\eta}{H} + P$$

$$P'_B = +\rho_B U^2 \frac{\eta}{H} + P$$

Investigation of Behavior of Interface with non-Stationary Flow Field using "Bernoulli's Principle"



>Supposing the Small Element Locates on Interface, the Motion of Elem. is Evaluated.

> Forces Affects on Elem.

Pressure:

From Upside, Elem. is Pushed Downward with

$$P'_U = -\rho_U U^2 \frac{\eta}{H} + P$$

From Underside, Elem. is Pushed Upward with

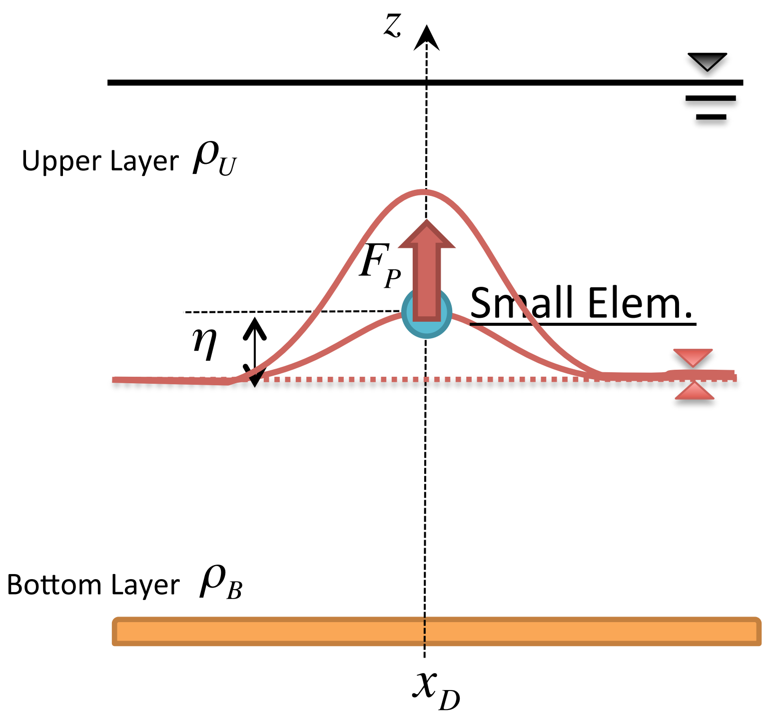
$$P'_B = +\rho_B U^2 \frac{\eta}{H} + P$$

Substituting

Net Pressure Force: $F_P = -\frac{\partial P}{\partial z} = \left(-P'_U + P'_B\right) / H$

$$\therefore F_P = \left(\rho_B + \rho_U\right) U^2 \frac{\eta}{H^2}$$

Investigation of Behavior of Interface with non-Stationary Flow Field using "Bernoulli's Principle"



Net Pressure Force Acting on Elem.:

$$\therefore F_P = (\rho_B + \rho_U) U^2 \frac{\eta}{H^2}$$

When Interface Displaced Upward,

> Initial Displacement is Positive $\eta > 0$.

> Pressure Force is also Positive.

$$\therefore F_P = (\rho_B + \rho_U) U^2 \frac{\eta}{H^2} > 0$$

Pressure Force Directs Upward

&

It Acts to Enlarge the Displacement

※ If there is not Density Stratification & non-Stationary Water Flows Exist, Displacement of Interface is Growing Endlessly.

※ In other word, Non-Stationary Water Flows is Naturally Unstable.

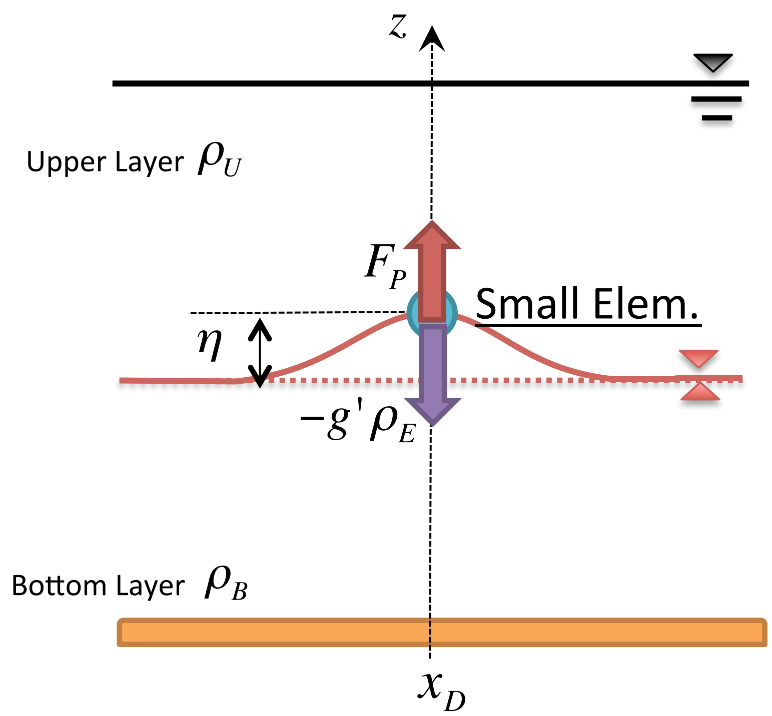
Investigation of Behavior of Interface with non-Stationary Flow Field

But, If There is Density Stratification,

In Addition to Pressure Force F_P,
Buoyancy Force is Also Affected on Elem.

$$\therefore -\rho_E g' = -\left(-\frac{g}{\rho_E}\frac{\partial \rho}{\partial z}\right)\times \eta$$

$$\rho_E = \frac{1}{2}(\rho_B + \rho_U)$$:Density of Elem.



Equation of Element's Motion:

$$\rho_E \frac{d^2\eta}{dt^2} = F_P - \rho_E g'$$

$$\rho_E \frac{d^2\eta}{dt^2} = (\rho_B + \rho_U)U^2 \frac{\eta}{H^2} + \left(\frac{g}{\rho_E}\frac{\partial \rho}{\partial z}\right)\times \eta$$

$$\therefore \frac{d^2\eta}{dt^2} = \frac{2U^2}{H^2}(1 - R_i)\eta$$

$$R_i \equiv \left(-\frac{g}{\rho_E}\frac{\partial \rho}{\partial z}\right) \Big/ \left(\frac{U^2}{H^2}\right)$$: Richardson Number

Solution of this Differential Eq. is Easy
Theoretical Solution Depends on Value of "Ri"

Investigation of Behavior of Interface with non-Stationary Flow Field

Richardson Number: $R_i \equiv \left(-\frac{g}{\rho_E}\frac{\partial \rho}{\partial z}\right) \Big/ \left(\frac{U^2}{H^2}\right)$

If R_i is Large and $R_i >> 1$,

Equation of Element's Motion:

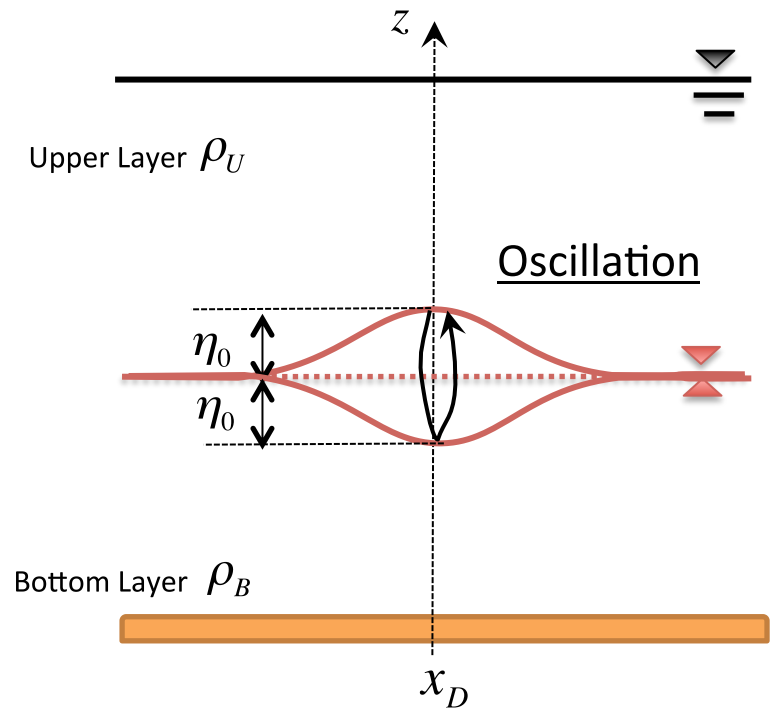
$$\frac{d^2\eta}{dt^2} = \frac{2U^2}{H^2}(1-R_i)\eta$$

Coefficient is Negative: $\frac{2U^2}{H^2}(1-R_i) < 0$

Solution is Oscillation with Constant Amplitude

$$\eta = \eta_0 \cos(\alpha \times t)$$

$$\alpha \equiv \sqrt{-\frac{2U^2}{H^2}(1-R_i)}$$



※ When R_i is Large ($R_i >> 1$), Even if Displacement Occurred Due to Something, Interface is Merely Oscillating & Stratification is Not Broken (Stable Stratification).

Investigation of Behavior of Interface with non-Stationary Flow Field

Richardson Number: $R_i \equiv \left(-\frac{1}{\rho_E}\frac{\partial \rho}{\partial z}\right) \Big/ \left(\frac{U^2}{H}\right)$

If R_i is Small and $R_i << 1$,

Equation of Element's Motion:

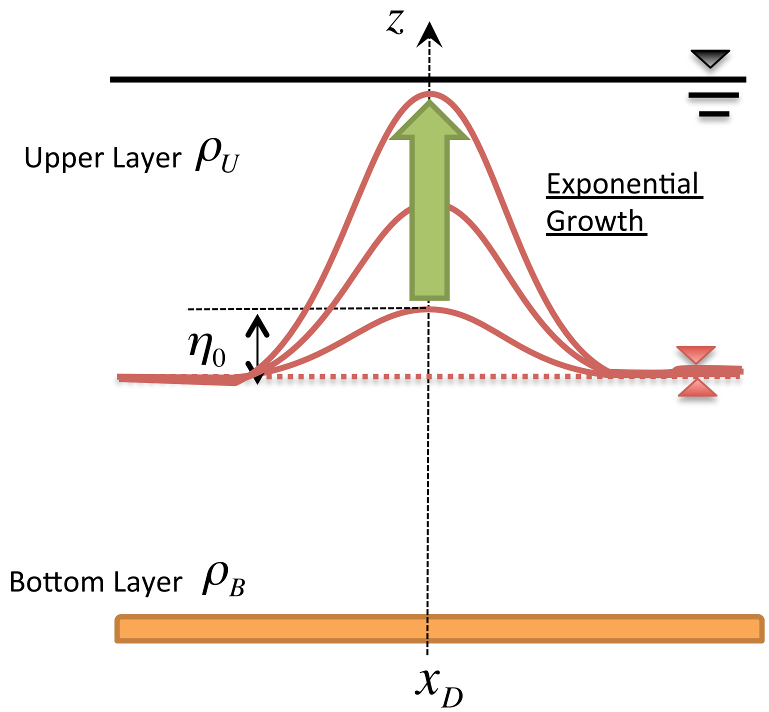
$$\frac{d^2\eta}{dt^2} = \frac{2U^2}{H^2}(1 - R_i)\eta$$

Coefficient is Positive: $\frac{2U^2}{H^2}(1 - R_i) > 0$

Solution is Given by Exponential Function

$$\eta = \eta_0 \exp(\alpha \times t)$$

$$\alpha \equiv \sqrt{\frac{2U^2}{H^2}(1 - R_i)}$$

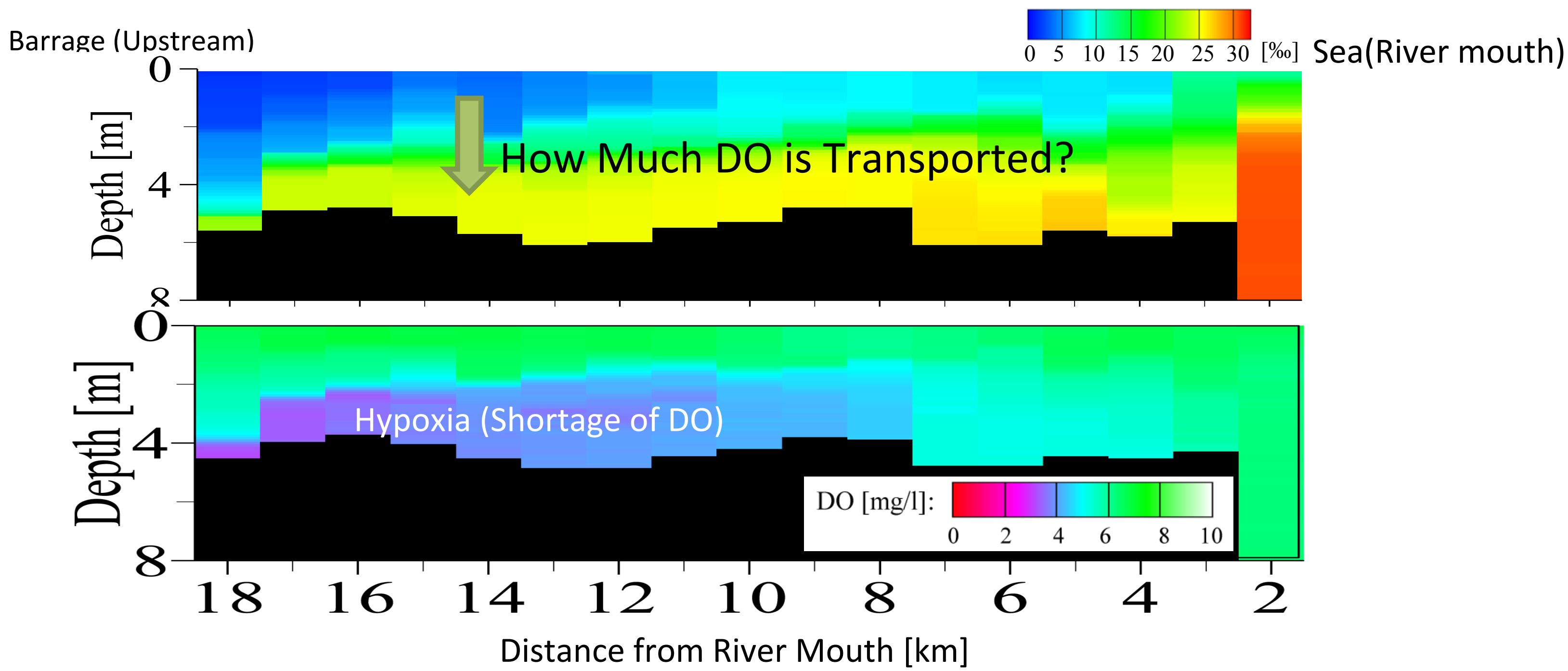


※ When R_i is Small ($R_i << 1$), if Once Displacement Occurred Due to Something, Interface Displaces Exponentially & Endlessly.

※ Finally, Stratification is Broken (Unstable Stratification).

“Ri” takes a roll in Environmental Hydraulics Research

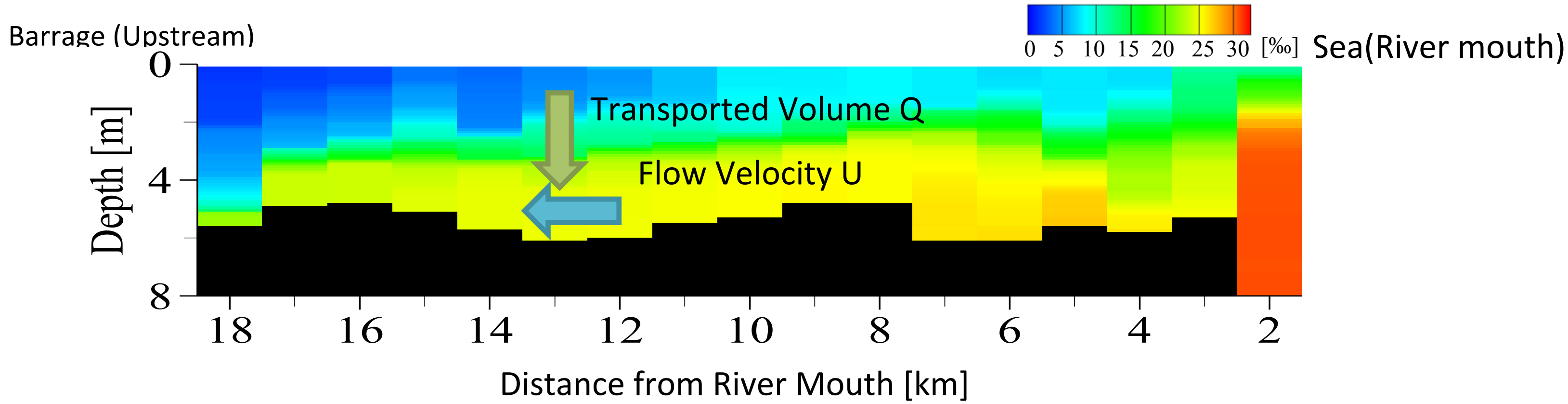
>Observed Salinity & DO Distribution Along the River Channel.



> In the “Salt Wedge”, there is the possibility Hypoxia Occurs.

> It is Important to Know “How much Substances (DO) can Transported Across the “Interface”.

“Ri” takes a roll in Environmental Hydraulics Research



- In Environmental Studies, Generally, Assumed Amount of the Transported Volume is in proportion to Velocity U & Efficiency of Transported Volume is Represented by “Entrainment Coefficient”.

$$Q = E \times U$$

E :Entrainment Coefficient

Large E : Upper and Bottom Layer Tends to be Well Mixed & Vertical Transportation Across the Interface is Large.

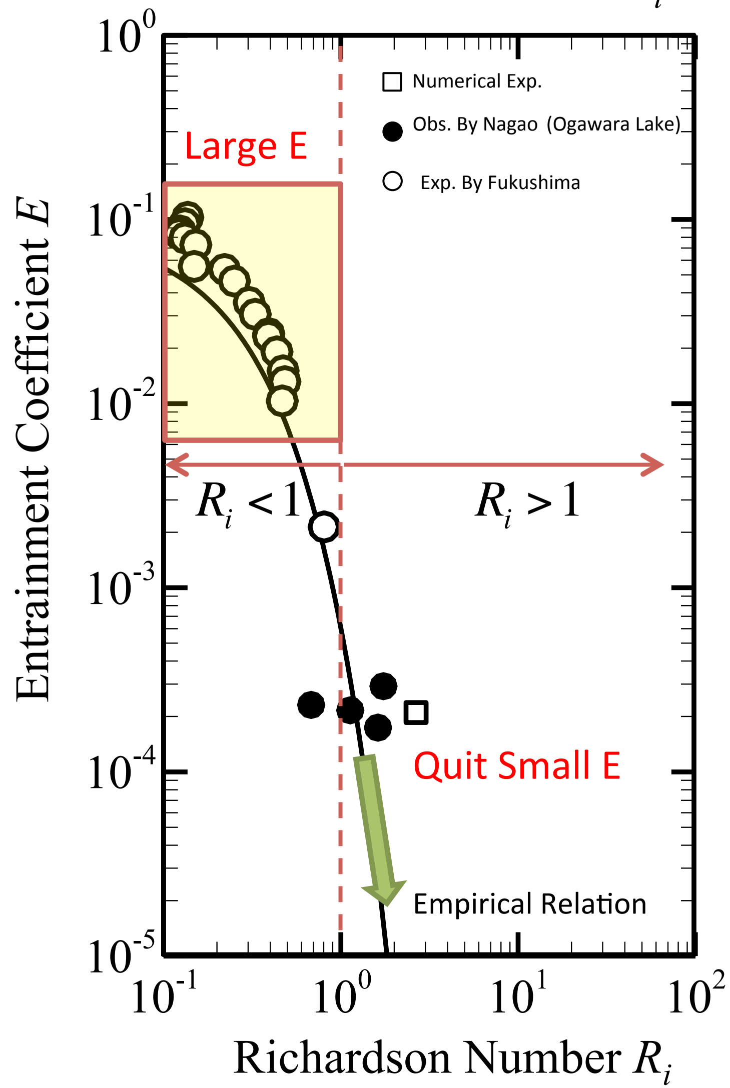
Small E : Upper and Bottom Layer Tends to be Not Mixed & Vertical Transportation Across the Interface is Small.

Actual Value of E is Evaluated Through "Flume Exp." and "Field Observation"

- Entrainment E Strongly Depends on Richardson Number R_i .
- When R_i is Smaller than "1", Entrainment is Large ($E > 0.01$) .
- When R_i is Larger than "1", Entrainment is Quite Small Value.

- Drastic Change of E at $R_i = 1$ can be Understood being Based on the Stability of Interface under non-stationary Flow That is Discussed Before.

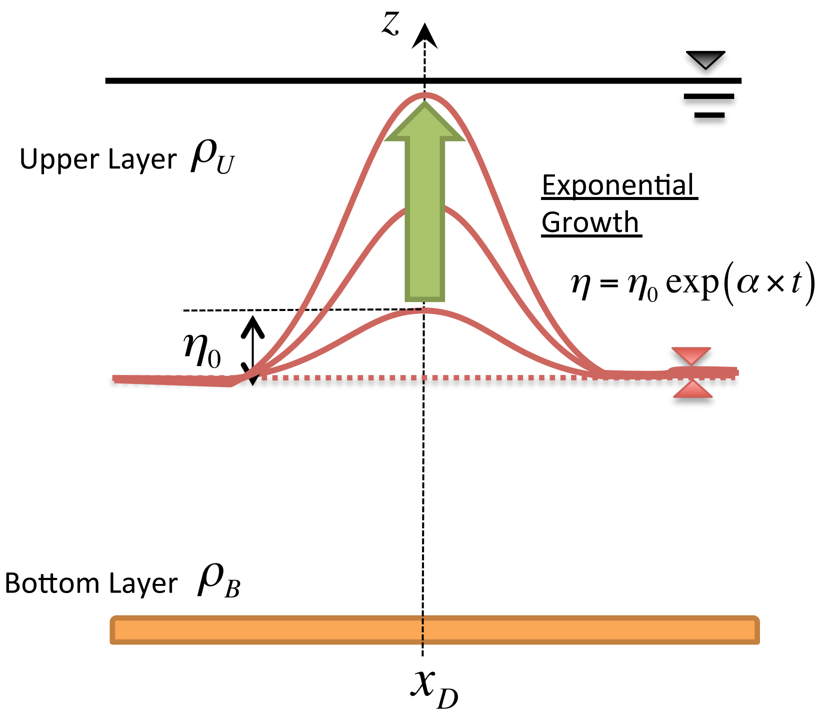
Relation Between E & R_i

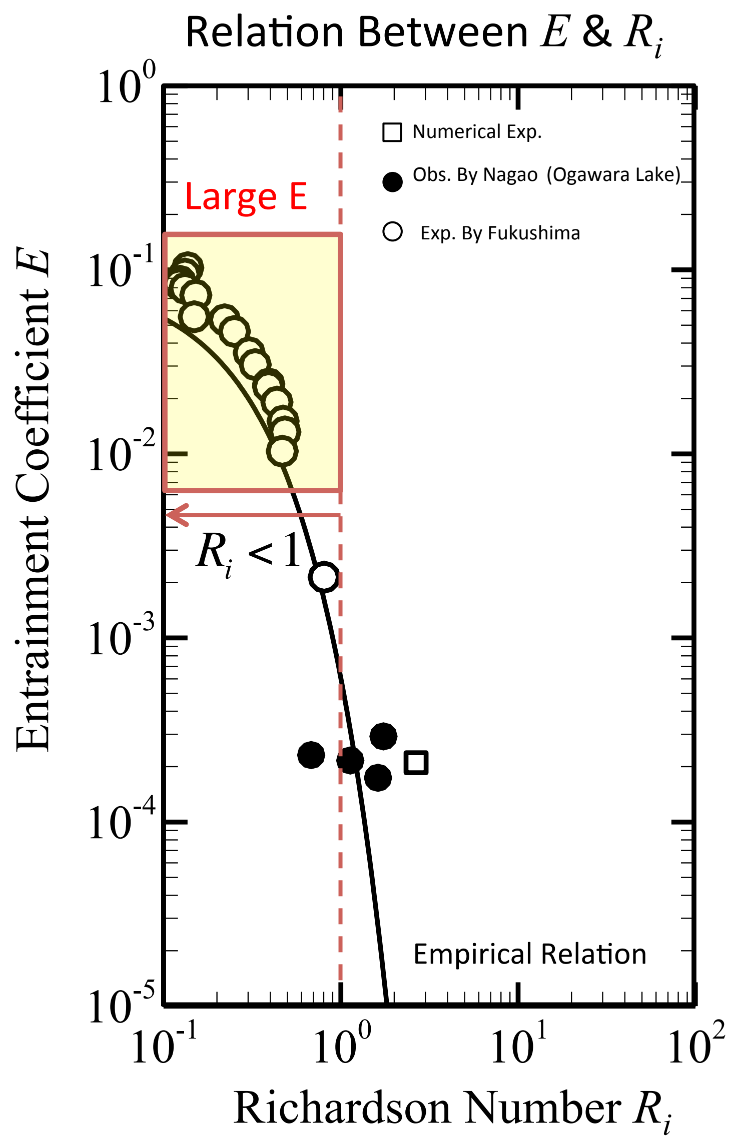


Actual Value of E is Evaluated Through "Flume Exp." and "Field Observation"

- When R_i is Smaller than "1",

  ⇨ Stratification is Unstable.





Actual Value of E is Evaluated Through "Flume Exp." and "Field Observation"

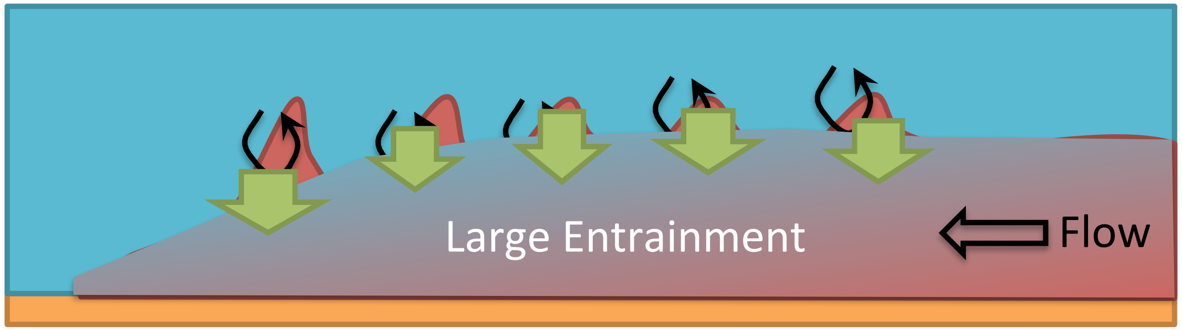
- When R_i is Smaller than "1",

  ⇨ Stratification is Unstable.

>Perturbation on Interface is Growing in Progressing.

>Finally, Stratification is Broken & Bottom Layer is Mixed with Upper Layer.

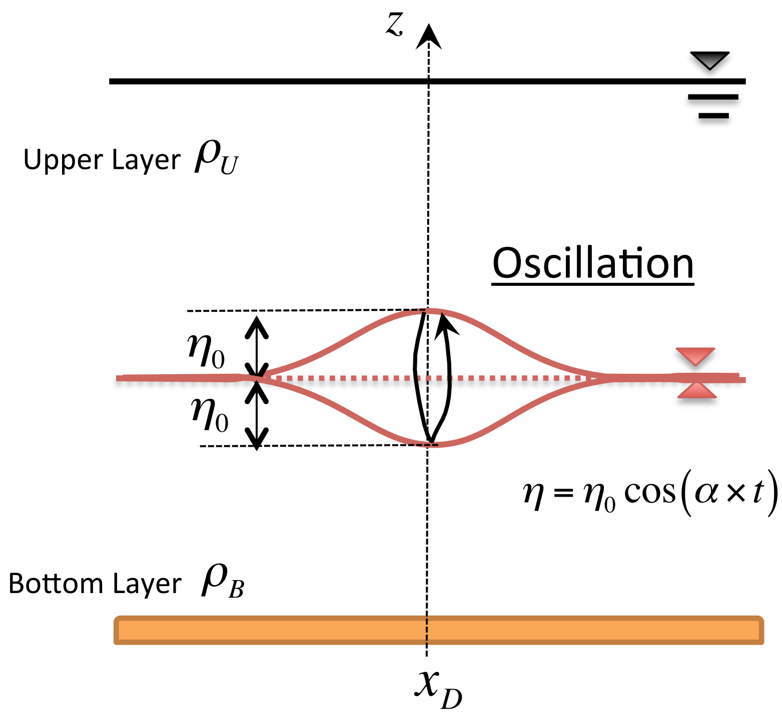
>As a Result, Transportation From the Upper Layer is Enhanced
& Large Entrainment is Observed.

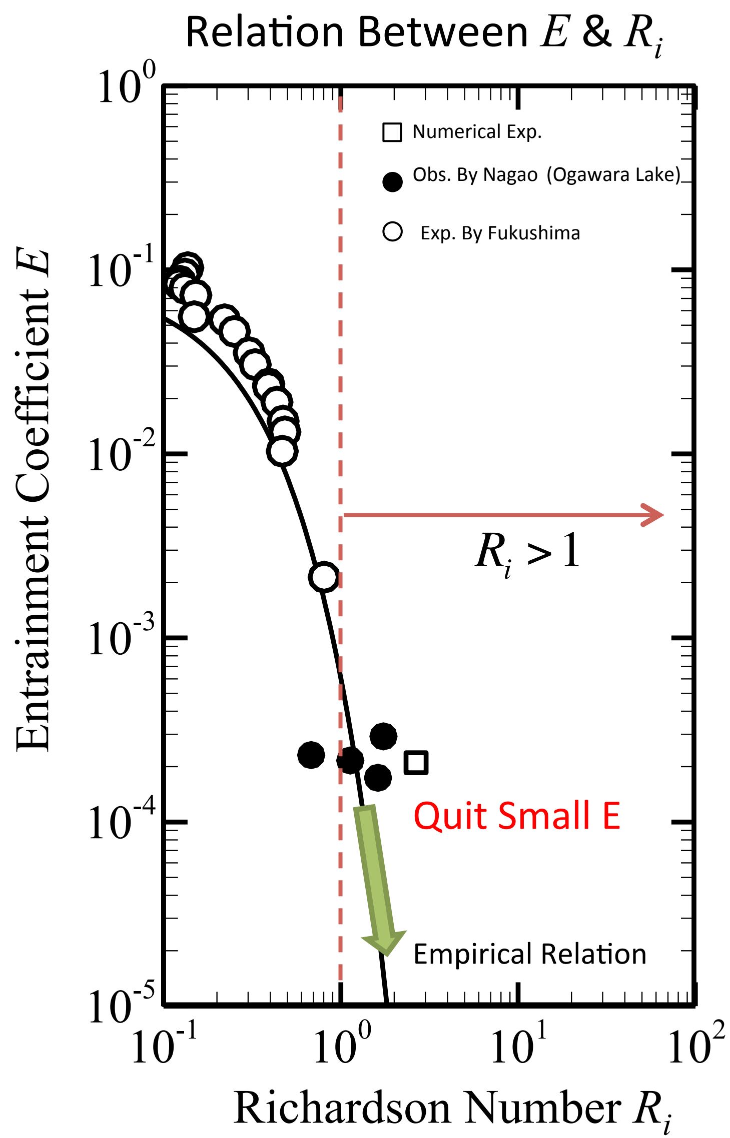


Actual Value of E is Evaluated Through "Flume Exp." and "Field Observation"

- When R_i is Larger than "1",

  ⇨ Stratification is Stable.





Actual Value of E is Evaluated Through “Flume Exp.” and “Field Observation”

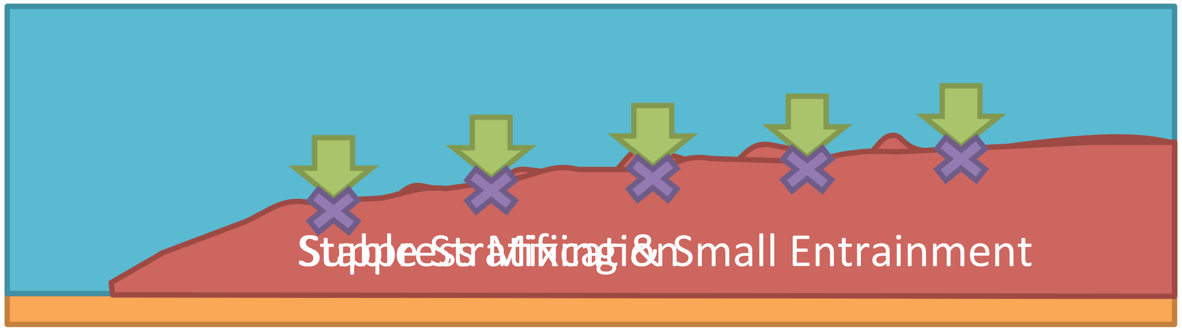
- When R_i is Larger than “1”,

  ⇨ Stratification is Stable.

>Even if Perturbation on Interface Occurred in Progressing,
Perturbation is not Growing.

> Stratification is Kept to be Stable & Mixing with Upper Layer is Still Suppressed.

>As a Result, Transportation From the Upper Layer is Reduced
& Small Entrainment is Observed.

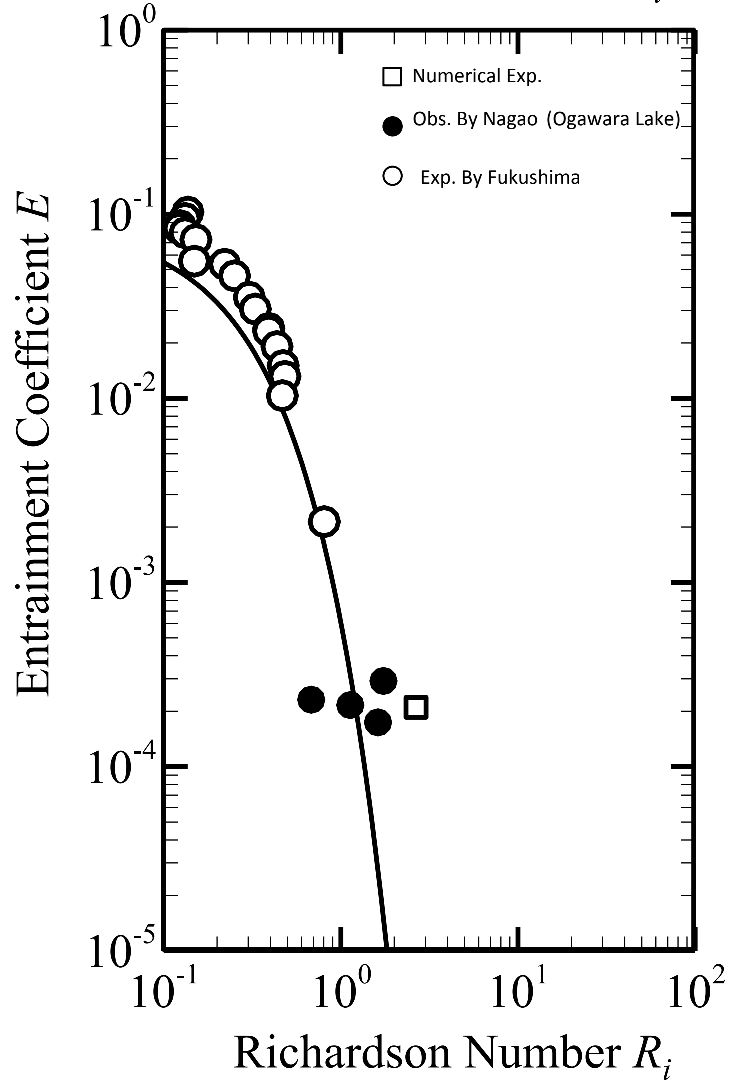


Under non-Stationary Flow Field,

Stability of Stratification & Transportation Across the Interface Strongly Depends on "Richardson Number".

So, "Richardson Number" Often Takes a Important Roll to Grasp the Water Environment with Density Stratification.

Relation Between E & R_i



Summary of Dynamics of Density Stratification Under Non-Stationary Flow

- In Actual Natural Water Environments, Horizontal Spatially Density Change Causes the Flows Named "Density Current".
- Most Typical Phenomena of "Density Current" is "Salt Wedge" Observed in "Estuary".
- We Investigated Dynamics & Stability of Density Stratification Based on "Bernoulli's Principle".
- Dynamics of Interface Depends on "Richardson Number".
- When "Richardson Number" is Large (Typically $Ri>1$) , Interface Oscillates Stably and Stratification is Kept to be Stable.
- When "Richardson Number" is Small (Typically $Ri<1$) , If Once Displacement / Perturbation Occurs on Interface, Displacement / Perturbation is Growing Exponentially and Endlessly. As a result, Stratification can not be Kept Stable.
- Transportation of Substances Across the Interface Relates to Stability of Density Stratification. The Amount of Transportation Changes According to "Ricardson Number".
- "Ricardson Number" can be an Essential Indicator to Grasp Fundamental States of "Water Environment with Density Stratification".